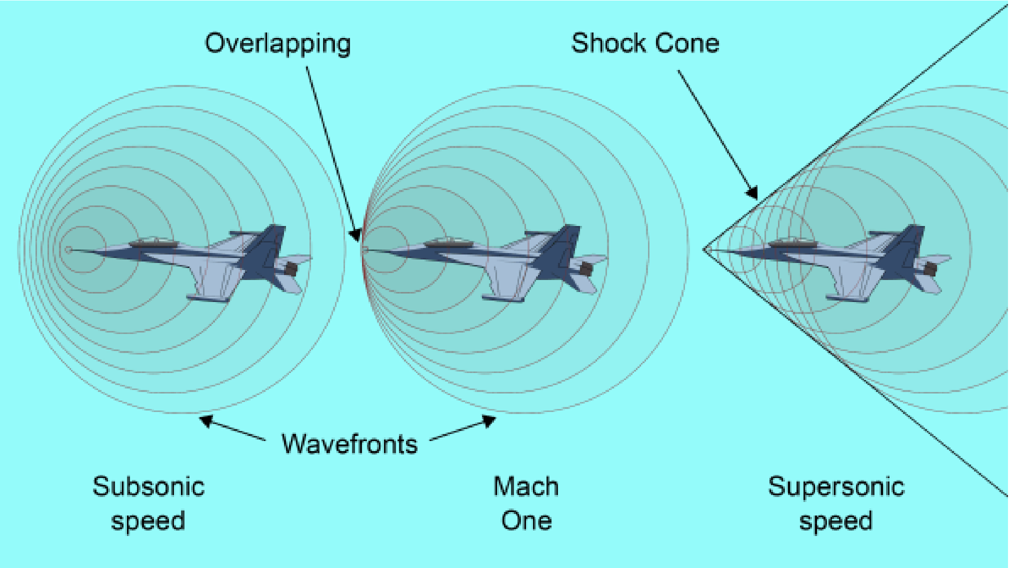
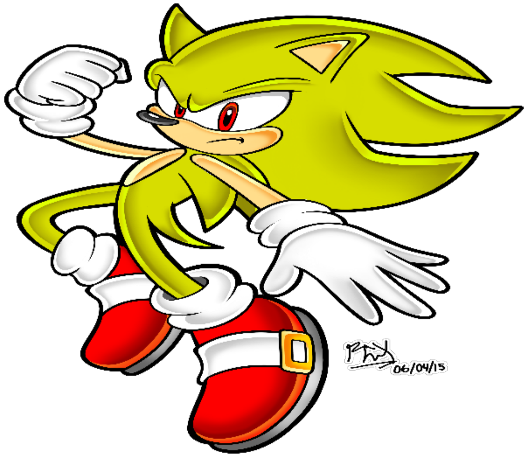
SUPERSONIC FLUID DYNAMICS
Ma > 0.2





BOOM VIDEO

Compressible Flow

Mach Number:

$$M = \frac{V}{c}$$

V: Velocity of Fluid

c: Speed of Sound

$c = \sqrt{kRT}$; R=Ideal gas constant (for air, R=287 N.m/kgK)

M < 0.3 incompressible flow

M < 1.0 subsonic flow

M @ 1.0 transonic flow

M = 1.0 sonic flow

M > 1.0 supersonic flow

M > 3.0 hypersonic flow

Class 18: Compressible Flow - Ideal gas law

Ideal/Perfect Gas Law (equation of state for an ideal gas):

Changes in gas density directly related to changes in pressure and temperature through the equation

$$\boxed{p = \rho R T}$$

, p is the pressure, ρ is the density, T is the absolute temperature and R is a gas constant.

$$\boxed{\mathrm{M} = \frac{\mathrm{V}}{\mathrm{C}}}$$

Mass flow rate:

$$\dot{m} = \rho A V = \frac{P}{RT} A V$$

$$\Rightarrow \dot{m} = \frac{P}{RT} A * M * C = \frac{P}{RT} A * M * \sqrt{kRT} = P * A * M * \sqrt{\frac{k}{RT}}$$

Class 18: Compressible Flow - Example

Example: **Air at Mach 1.25 passes through a circular channel 10 cm in diameter. The static pressure and temperature are 100kpa and 30°C respectively. Find the mass flow rate.**

Solution: **The mass flow rate can be calculated as**

$$\dot{m} = \rho AV = \frac{P}{RT}AV = P * A * M * \sqrt{\frac{k}{RT}}$$

$$\Rightarrow \dot{m} = 100kPka \times \left(\frac{\left(10^3\, N/m^3\right)}{1kPa} \right) \times \frac{\pi}{4}\left(0.01m\right)^2 \times 1.25\sqrt{\frac{1.4}{\left(287\, N.m/kg^o K\right)\left(273+30\right)^o K}}$$

$$\Rightarrow \dot{m} = 3.93\frac{kg}{s}$$

Class 18: Compressible Flow – different processes

- **Adiabatic Process**

An adiabatic process is one in which no heat is gained or lost by the system. The first law of thermodynamics with Q=0 shows that all the change in internal energy is in the form of work done. This puts a constraint on the heat engine process leading to the adiabatic condition. This condition can be used to derive the expression for the work done during an adiabatic process.

$$PV^k = \text{constant} = C;\ \text{k}=\text{C}_\text{p}/\text{C}_\text{V} \text{ ratio of specific heats.}$$

- **Reversible process**

A *reversible process* is a process that, after it has taken place, can be reversed and causes no change in either the system or its surroundings.

- **Isentropic flow**

An isentropic flow is a flow that is both adiabatic and reversible. That is, no energy is added to the flow, and no energy losses occur due to friction or dissipative effects.

Class 18: Compressible Flow – different processes

Relationship between temperature, density and pressure for the isentropic flow of an ideal gas:

$$\left(\frac{T}{T_0}\right)^{k/k-1} = \left(\frac{\rho}{\rho_0}\right)^{k} = \left(\frac{p}{p_0}\right)$$

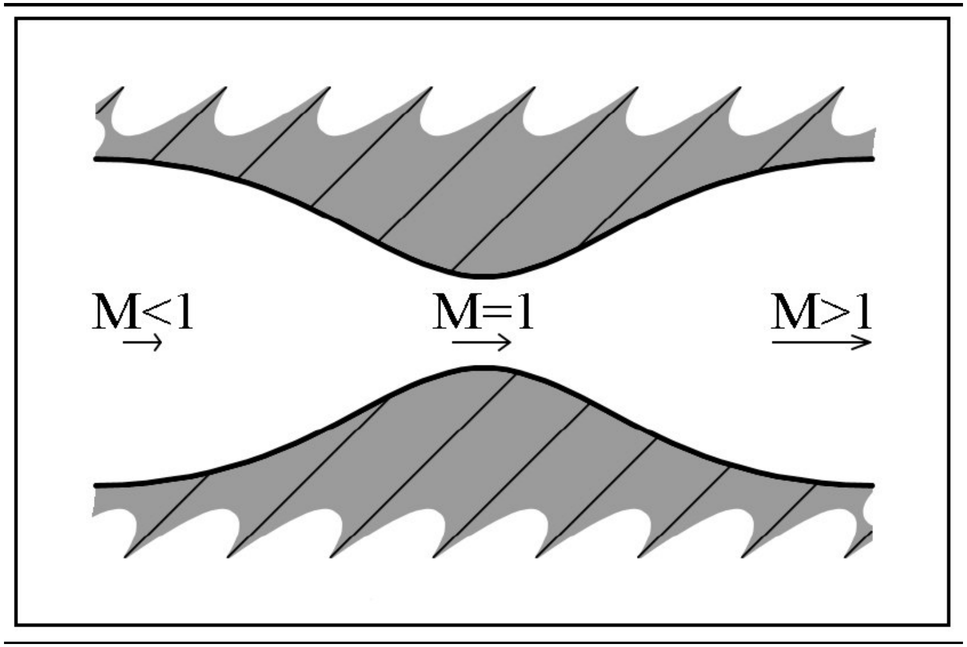
$$\Rightarrow \quad \frac{p}{\rho^k} = \text{constant}$$

For isentropic flow:

$$c = \sqrt{\left(\frac{dp}{d\rho}\right)_s} = \text{speed of sound}$$

Class 18: Compressible Flow → Converging-Diverging Nozzle

Experiment: Converging-diverging nozzle



Subsonic → **Sonic** → **Supersonic Flow**

Class 18: Effect of Variations in Flow Cross-sectional Area

Newtons 2nd law applied to the inviscid and steady flow (Bernoulli):

$$dp + 1/2\,\rho d\left(V^2\right) + \gamma dz = 0$$

$$\Rightarrow dp + 1/2\,\rho d\left(V^2\right) = 0;\ \text{for ideal gas P. E. term dropped}$$

$$\Rightarrow \frac{dp}{\rho V^2} = -\frac{dV}{V}$$

$$\dot{m} = \rho A V = \text{constant}$$

$$\Rightarrow \ln\rho + \ln A + \ln V = 0$$

$$\Rightarrow \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$$

$$\Rightarrow -\frac{dV}{V} = \frac{d\rho}{\rho} + \frac{dA}{A}$$

$$\Rightarrow \frac{dp}{\rho V^2}\left(1 - \frac{V^2}{dp/d\rho}\right) = \frac{dA}{A}$$

$$c = \sqrt{\left(\frac{dp}{d\rho}\right)};\ \mathbf{M} = \frac{\mathbf{V}}{\mathbf{c}}$$

$$c^2 = \left(\frac{dp}{d\rho}\right) = \frac{\mathbf{V}^2}{M^2};\ or$$

$$\frac{\mathbf{V}^2}{\frac{dp}{d\rho}} = M^2$$

$$\frac{dp}{\rho V^2}\left(1 - M^2\right) = \frac{dA}{A}$$

$$\Rightarrow dp\left(1 - M^2\right) = \rho V^2 \frac{dA}{A}$$

Pressure, Mach #, density and velocity are correlated with AREA!!!

Class 18: Effect of Variations in Flow Cross-sectional Area ..cont

$$\frac{dp}{\rho V^2}\left(1-M^2\right)=\frac{dA}{A}$$

$$\Rightarrow dp\left(1-M^2\right)=\rho V^2\frac{dA}{A};but$$

$$\frac{dp}{\rho V^2}=-\frac{dV}{V};so$$

$$\frac{dV}{V}=-\frac{dA}{A}\frac{1}{(1-M^2)}$$

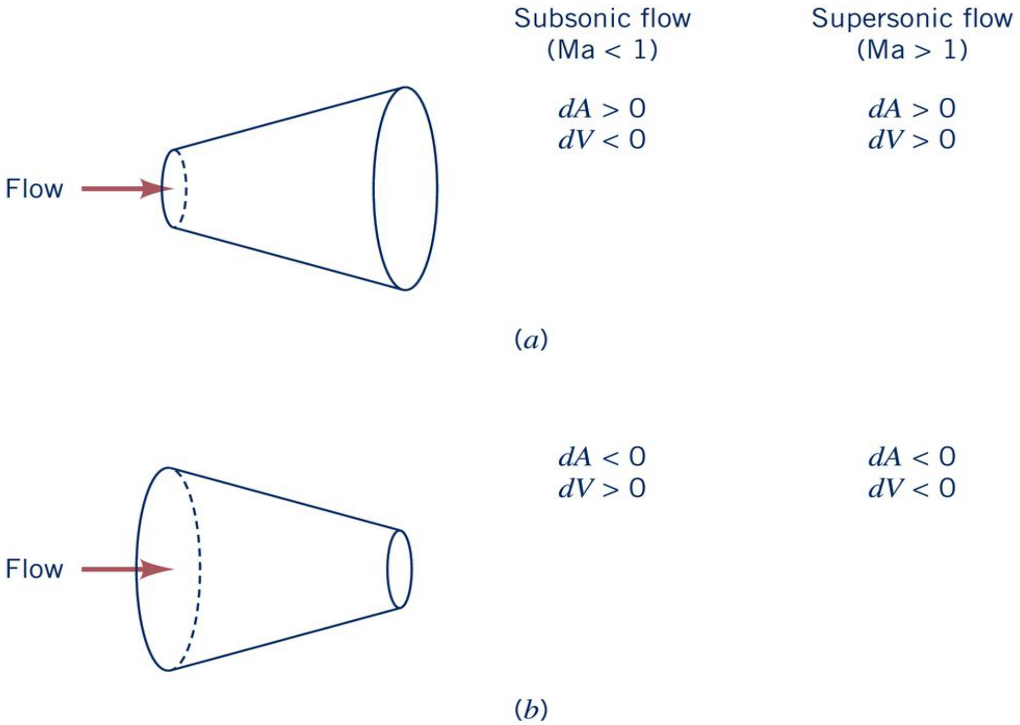


FIGURE 11.5

Class 18: Effect of Variations in Flow Cross-sectional Area

How Mach number influences Temperature, Pressure and Density of the fluid?

$$"t" \equiv \text{TOTAL OR STAGNATION}$$

$$\frac{T_t}{T} = 1 + \frac{k-1}{2} M^2; \qquad k = \frac{c_p}{c_v}$$

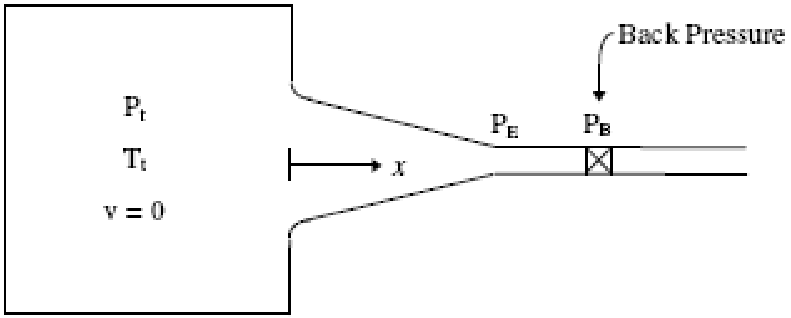
$$\frac{P_t}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$$

$$\frac{\rho_t}{\rho} = \left[1 + \frac{k-1}{2} M^2\right]^{1/(k-1)}$$

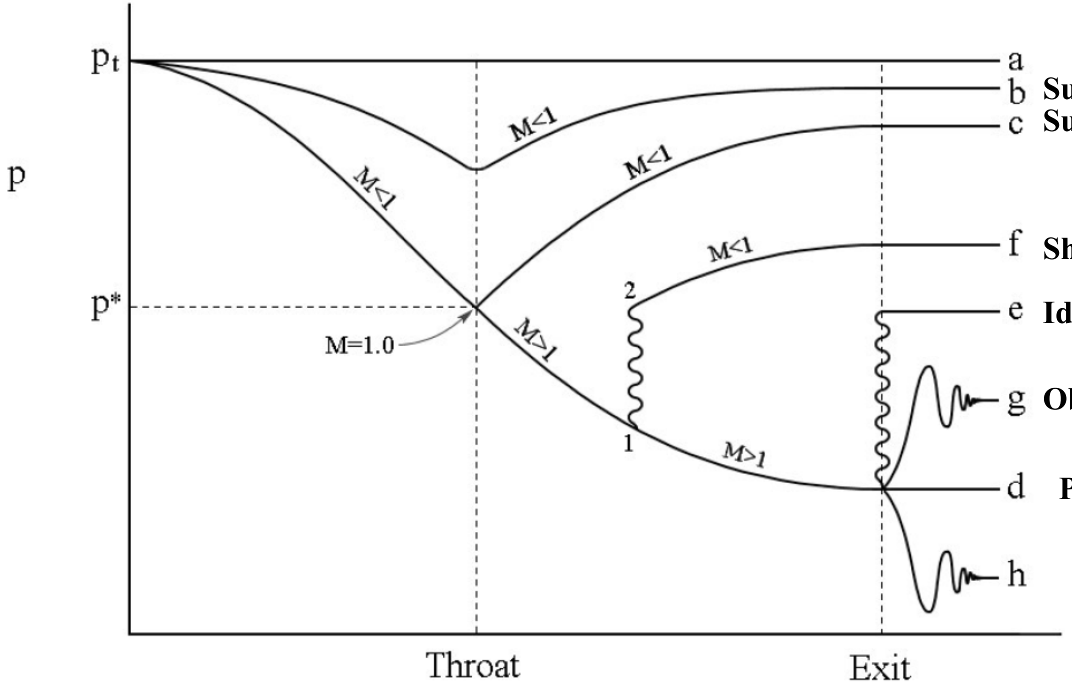
Temperature, Pressure and density can be tabulated for a given value of k (for air, k=1.4).

Class 18: Compressible Flow → Converging-Diverging Nozzle

Effect of Back Pressure on Flow Pattern: Shockwave and expansion



- **Normal Shockwave** (abrupt rise or drop of pressure)
- **Ideally/Perfectly Expanded**
- **Over Expanded** (pressure rises at the duct exit)
- **Under Expanded** (pressure drops at the duct exit)
- **Oblique Shockwave** (less abrupt rise or drop of pressure)



b Subsonic all the way

c Subsonic-sonic-subsonic

f Shockwave half-way

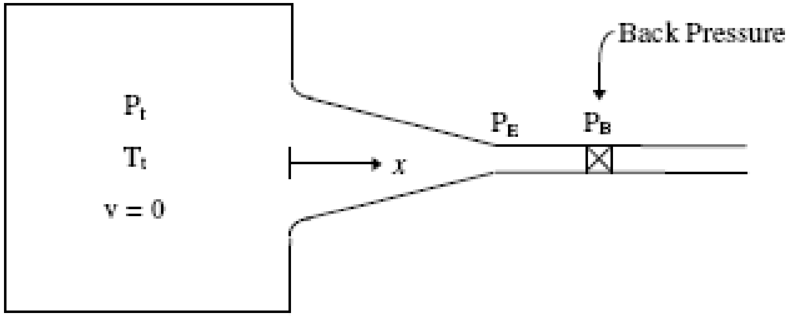
e Ideally Expanded

g Oblique shockwave

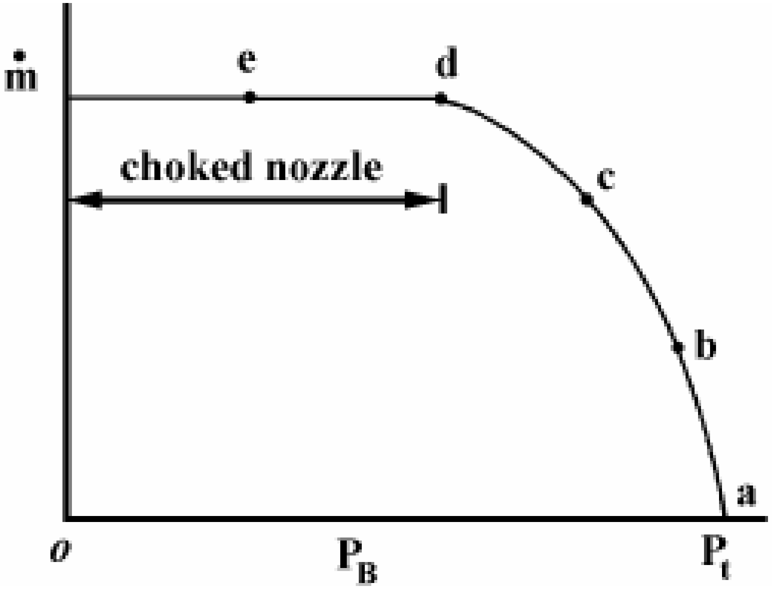
d Perfectly Expanded

Class 18: Compressible Flow → Converging-Diverging Nozzle

Choked Flow/ Unchoked Flow:

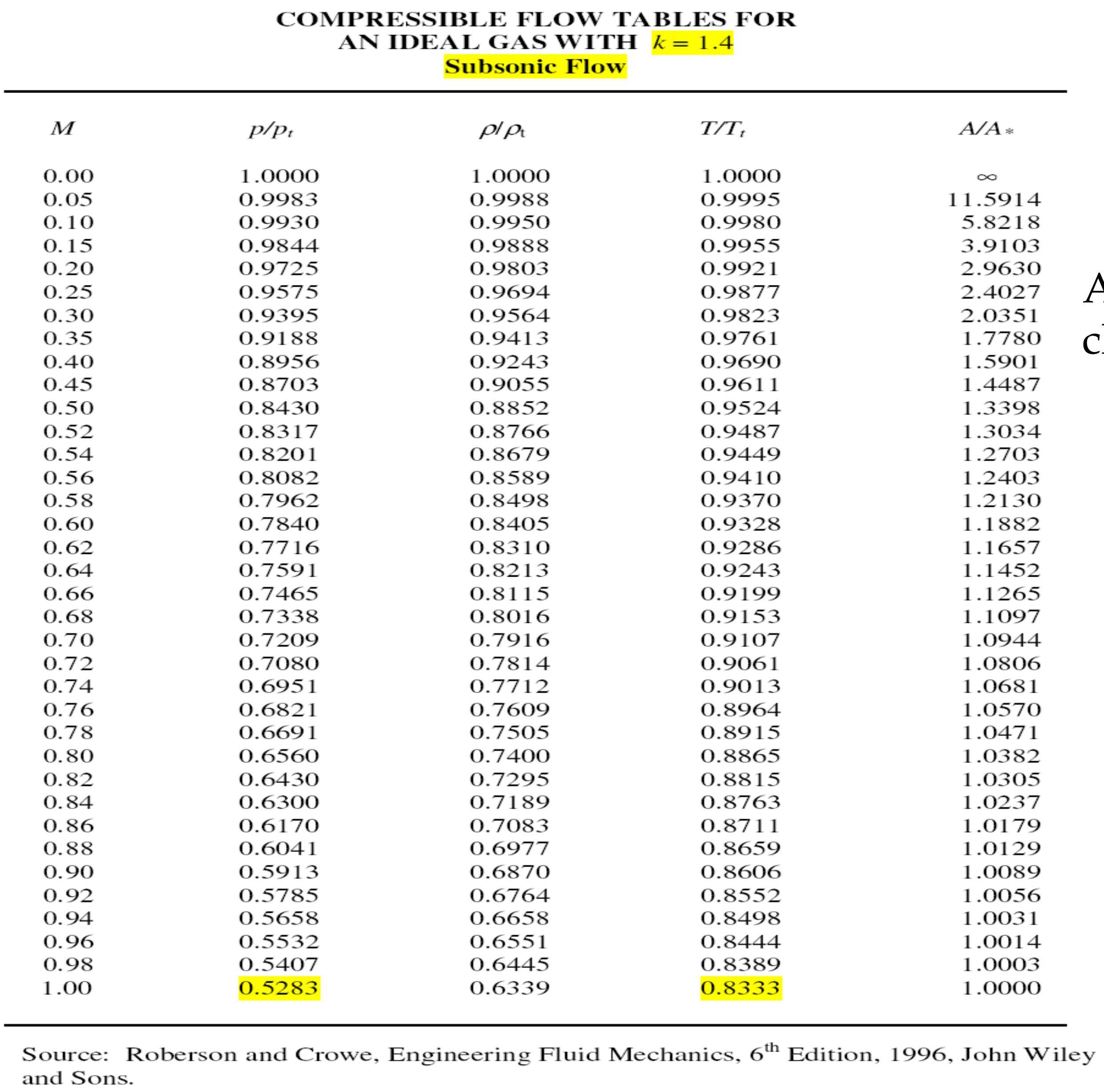


Shockwave: Each abrupt pressure rise within and at the exit of the flow passage occurs across a very thin discontinuity in the flow called a **Normal Shockwave**.



Choked Flow: Choked flow occurs when the Mach number is 1.0 at the minimum cross-sectional area.

Compressible Flow - Table



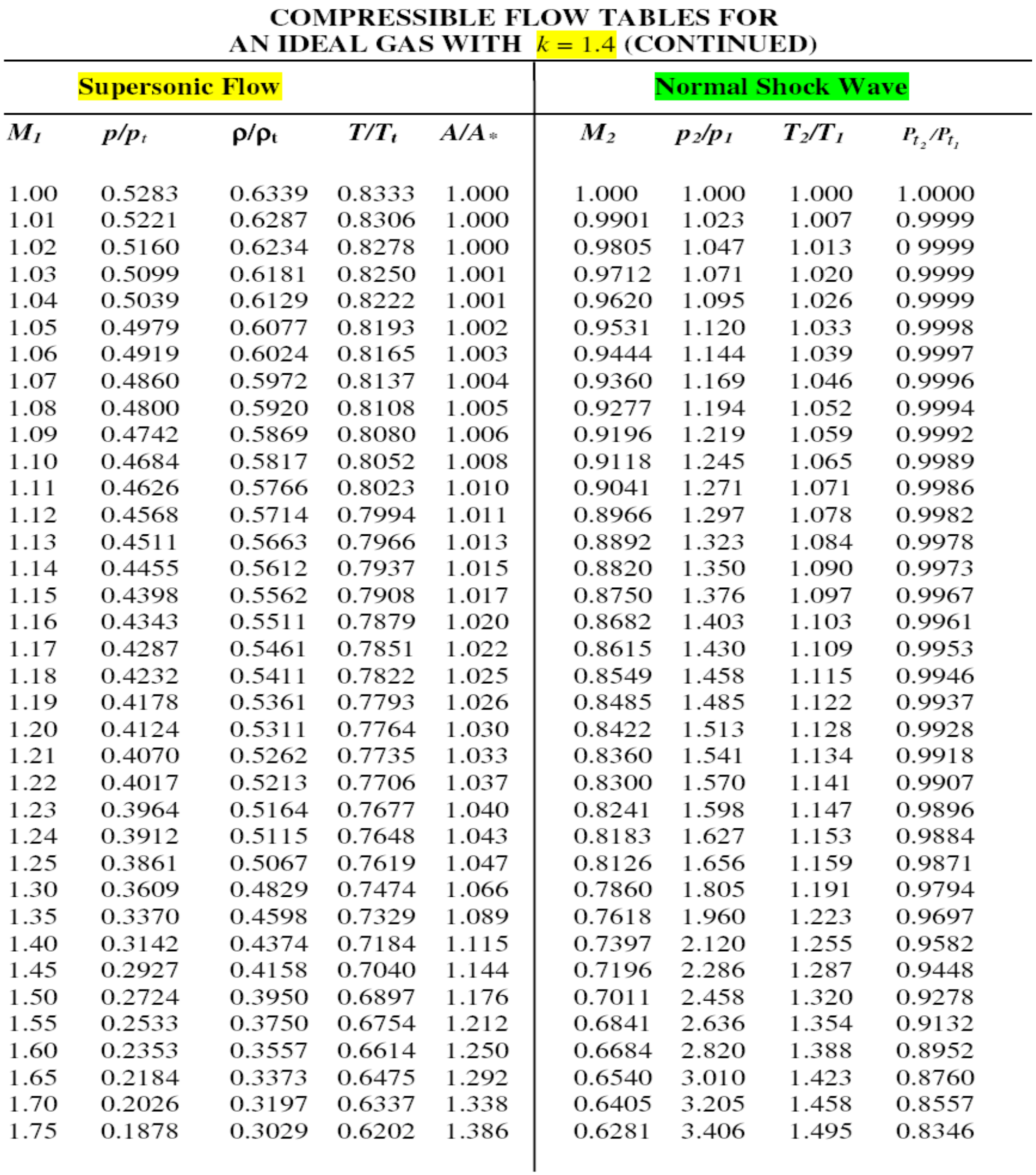
COMPRESSIBLE FLOW TABLES FOR
AN IDEAL GAS WITH $k = 1.4$
Subsonic Flow

M	p/p_t	ρ/ρ_t	T/T_t	A/A_*
0.00	1.0000	1.0000	1.0000	∞
0.05	0.9983	0.9988	0.9995	11.5914
0.10	0.9930	0.9950	0.9980	5.8218
0.15	0.9844	0.9888	0.9955	3.9103
0.20	0.9725	0.9803	0.9921	2.9630
0.25	0.9575	0.9694	0.9877	2.4027
0.30	0.9395	0.9564	0.9823	2.0351
0.35	0.9188	0.9413	0.9761	1.7780
0.40	0.8956	0.9243	0.9690	1.5901
0.45	0.8703	0.9055	0.9611	1.4487
0.50	0.8430	0.8852	0.9524	1.3398
0.52	0.8317	0.8766	0.9487	1.3034
0.54	0.8201	0.8679	0.9449	1.2703
0.56	0.8082	0.8589	0.9410	1.2403
0.58	0.7962	0.8498	0.9370	1.2130
0.60	0.7840	0.8405	0.9328	1.1882
0.62	0.7716	0.8310	0.9286	1.1657
0.64	0.7591	0.8213	0.9243	1.1452
0.66	0.7465	0.8115	0.9199	1.1265
0.68	0.7338	0.8016	0.9153	1.1097
0.70	0.7209	0.7916	0.9107	1.0944
0.72	0.7080	0.7814	0.9061	1.0806
0.74	0.6951	0.7712	0.9013	1.0681
0.76	0.6821	0.7609	0.8964	1.0570
0.78	0.6691	0.7505	0.8915	1.0471
0.80	0.6560	0.7400	0.8865	1.0382
0.82	0.6430	0.7295	0.8815	1.0305
0.84	0.6300	0.7189	0.8763	1.0237
0.86	0.6170	0.7083	0.8711	1.0179
0.88	0.6041	0.6977	0.8659	1.0129
0.90	0.5913	0.6870	0.8606	1.0089
0.92	0.5785	0.6764	0.8552	1.0056
0.94	0.5658	0.6658	0.8498	1.0031
0.96	0.5532	0.6551	0.8444	1.0014
0.98	0.5407	0.6445	0.8389	1.0003
1.00	0.5283	0.6339	0.8333	1.0000

Source: Roberson and Crowe, Engineering Fluid Mechanics, 6th Edition, 1996, John Wiley and Sons.

A* is critical area for choked flow at throat.

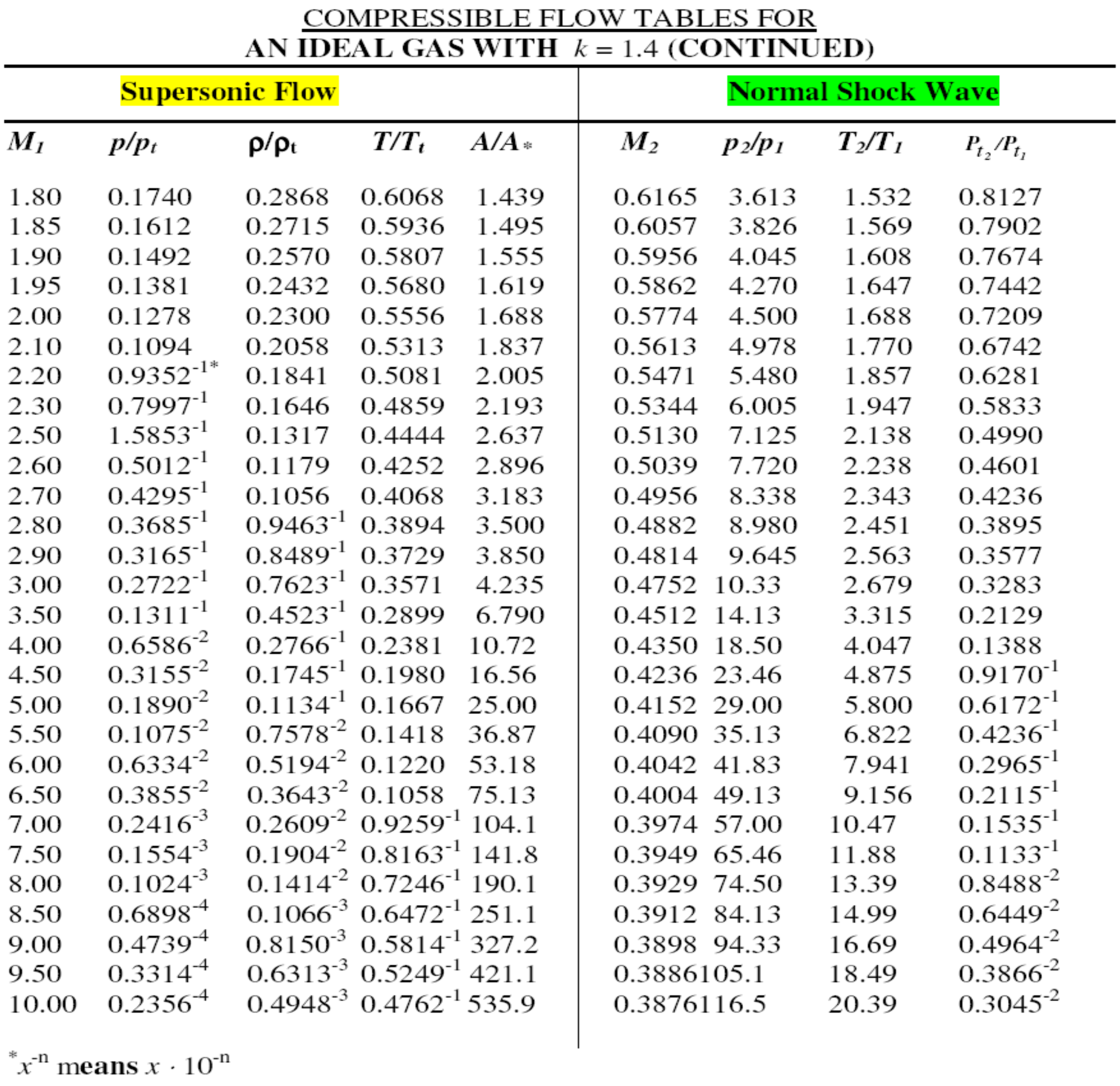
Compressible Flow - Table



COMPRESSIBLE FLOW TABLES FOR
AN IDEAL GAS WITH $k = 1.4$ (CONTINUED)

Supersonic Flow					**Normal Shock Wave**			
M_1	p/p_t	ρ/ρ_t	T/T_t	A/A_*	M_2	p_2/p_1	T_2/T_1	p_{t_2}/p_{t_1}
1.00	0.5283	0.6339	0.8333	1.000	1.000	1.000	1.000	1.0000
1.01	0.5221	0.6287	0.8306	1.000	0.9901	1.023	1.007	0.9999
1.02	0.5160	0.6234	0.8278	1.000	0.9805	1.047	1.013	0 9999
1.03	0.5099	0.6181	0.8250	1.001	0.9712	1.071	1.020	0.9999
1.04	0.5039	0.6129	0.8222	1.001	0.9620	1.095	1.026	0.9999
1.05	0.4979	0.6077	0.8193	1.002	0.9531	1.120	1.033	0.9998
1.06	0.4919	0.6024	0.8165	1.003	0.9444	1.144	1.039	0.9997
1.07	0.4860	0.5972	0.8137	1.004	0.9360	1.169	1.046	0.9996
1.08	0.4800	0.5920	0.8108	1.005	0.9277	1.194	1.052	0.9994
1.09	0.4742	0.5869	0.8080	1.006	0.9196	1.219	1.059	0.9992
1.10	0.4684	0.5817	0.8052	1.008	0.9118	1.245	1.065	0.9989
1.11	0.4626	0.5766	0.8023	1.010	0.9041	1.271	1.071	0.9986
1.12	0.4568	0.5714	0.7994	1.011	0.8966	1.297	1.078	0.9982
1.13	0.4511	0.5663	0.7966	1.013	0.8892	1.323	1.084	0.9978
1.14	0.4455	0.5612	0.7937	1.015	0.8820	1.350	1.090	0.9973
1.15	0.4398	0.5562	0.7908	1.017	0.8750	1.376	1.097	0.9967
1.16	0.4343	0.5511	0.7879	1.020	0.8682	1.403	1.103	0.9961
1.17	0.4287	0.5461	0.7851	1.022	0.8615	1.430	1.109	0.9953
1.18	0.4232	0.5411	0.7822	1.025	0.8549	1.458	1.115	0.9946
1.19	0.4178	0.5361	0.7793	1.026	0.8485	1.485	1.122	0.9937
1.20	0.4124	0.5311	0.7764	1.030	0.8422	1.513	1.128	0.9928
1.21	0.4070	0.5262	0.7735	1.033	0.8360	1.541	1.134	0.9918
1.22	0.4017	0.5213	0.7706	1.037	0.8300	1.570	1.141	0.9907
1.23	0.3964	0.5164	0.7677	1.040	0.8241	1.598	1.147	0.9896
1.24	0.3912	0.5115	0.7648	1.043	0.8183	1.627	1.153	0.9884
1.25	0.3861	0.5067	0.7619	1.047	0.8126	1.656	1.159	0.9871
1.30	0.3609	0.4829	0.7474	1.066	0.7860	1.805	1.191	0.9794
1.35	0.3370	0.4598	0.7329	1.089	0.7618	1.960	1.223	0.9697
1.40	0.3142	0.4374	0.7184	1.115	0.7397	2.120	1.255	0.9582
1.45	0.2927	0.4158	0.7040	1.144	0.7196	2.286	1.287	0.9448
1.50	0.2724	0.3950	0.6897	1.176	0.7011	2.458	1.320	0.9278
1.55	0.2533	0.3750	0.6754	1.212	0.6841	2.636	1.354	0.9132
1.60	0.2353	0.3557	0.6614	1.250	0.6684	2.820	1.388	0.8952
1.65	0.2184	0.3373	0.6475	1.292	0.6540	3.010	1.423	0.8760
1.70	0.2026	0.3197	0.6337	1.338	0.6405	3.205	1.458	0.8557
1.75	0.1878	0.3029	0.6202	1.386	0.6281	3.406	1.495	0.8346

Compressible Flow - Table

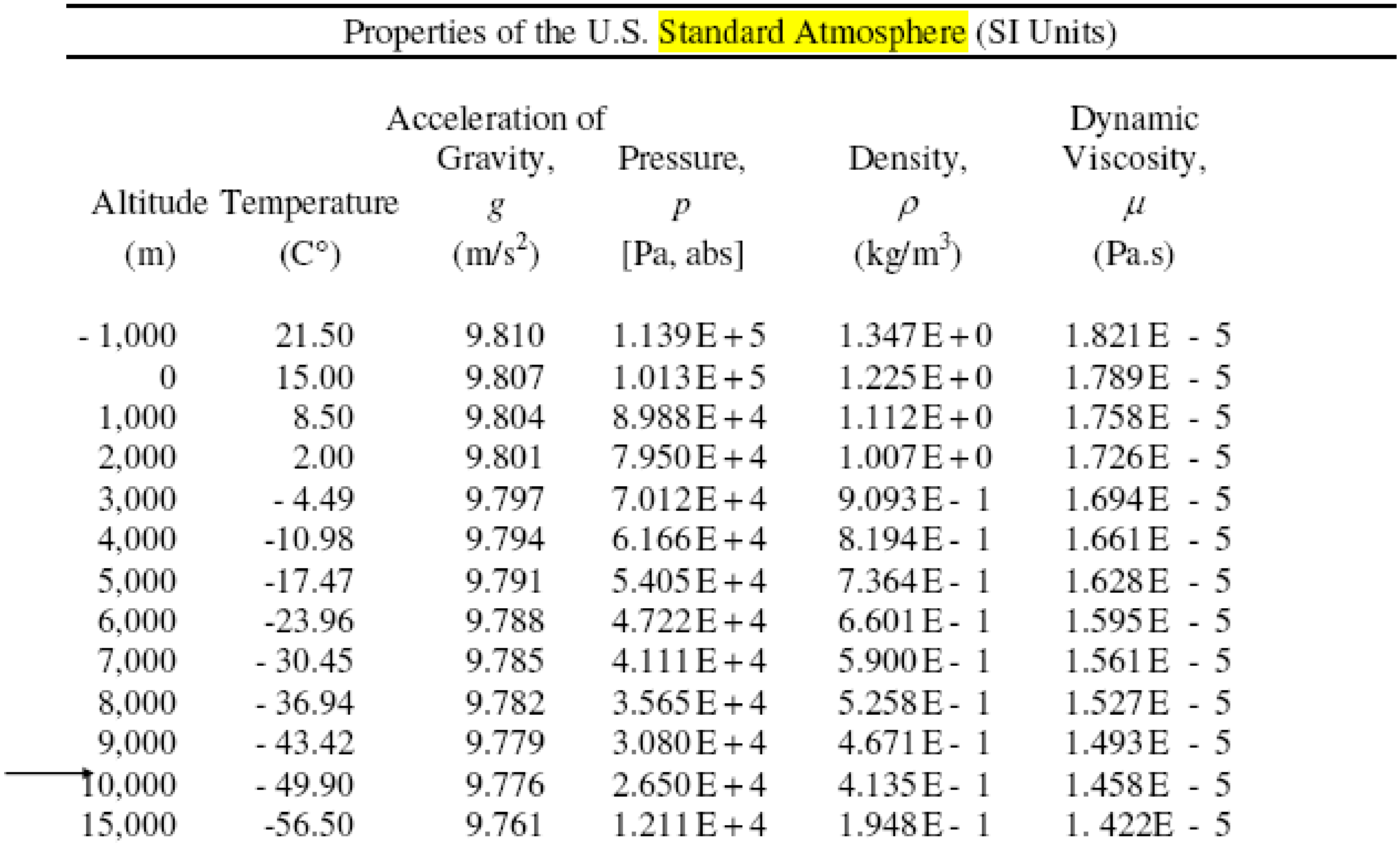


COMPRESSIBLE FLOW TABLES FOR
AN IDEAL GAS WITH $k = 1.4$ (CONTINUED)

Supersonic Flow					**Normal Shock Wave**			
M_1	p/p_t	ρ/ρ_t	T/T_t	A/A_*	M_2	p_2/p_1	T_2/T_1	P_{t_2}/P_{t_1}
1.80	0.1740	0.2868	0.6068	1.439	0.6165	3.613	1.532	0.8127
1.85	0.1612	0.2715	0.5936	1.495	0.6057	3.826	1.569	0.7902
1.90	0.1492	0.2570	0.5807	1.555	0.5956	4.045	1.608	0.7674
1.95	0.1381	0.2432	0.5680	1.619	0.5862	4.270	1.647	0.7442
2.00	0.1278	0.2300	0.5556	1.688	0.5774	4.500	1.688	0.7209
2.10	0.1094	0.2058	0.5313	1.837	0.5613	4.978	1.770	0.6742
2.20	0.9352^{-1*}	0.1841	0.5081	2.005	0.5471	5.480	1.857	0.6281
2.30	0.7997^{-1}	0.1646	0.4859	2.193	0.5344	6.005	1.947	0.5833
2.50	1.5853^{-1}	0.1317	0.4444	2.637	0.5130	7.125	2.138	0.4990
2.60	0.5012^{-1}	0.1179	0.4252	2.896	0.5039	7.720	2.238	0.4601
2.70	0.4295^{-1}	0.1056	0.4068	3.183	0.4956	8.338	2.343	0.4236
2.80	0.3685^{-1}	0.9463^{-1}	0.3894	3.500	0.4882	8.980	2.451	0.3895
2.90	0.3165^{-1}	0.8489^{-1}	0.3729	3.850	0.4814	9.645	2.563	0.3577
3.00	0.2722^{-1}	0.7623^{-1}	0.3571	4.235	0.4752	10.33	2.679	0.3283
3.50	0.1311^{-1}	0.4523^{-1}	0.2899	6.790	0.4512	14.13	3.315	0.2129
4.00	0.6586^{-2}	0.2766^{-1}	0.2381	10.72	0.4350	18.50	4.047	0.1388
4.50	0.3155^{-2}	0.1745^{-1}	0.1980	16.56	0.4236	23.46	4.875	0.9170^{-1}
5.00	0.1890^{-2}	0.1134^{-1}	0.1667	25.00	0.4152	29.00	5.800	0.6172^{-1}
5.50	0.1075^{-2}	0.7578^{-2}	0.1418	36.87	0.4090	35.13	6.822	0.4236^{-1}
6.00	0.6334^{-2}	0.5194^{-2}	0.1220	53.18	0.4042	41.83	7.941	0.2965^{-1}
6.50	0.3855^{-2}	0.3643^{-2}	0.1058	75.13	0.4004	49.13	9.156	0.2115^{-1}
7.00	0.2416^{-3}	0.2609^{-2}	0.9259^{-1}	104.1	0.3974	57.00	10.47	0.1535^{-1}
7.50	0.1554^{-3}	0.1904^{-2}	0.8163^{-1}	141.8	0.3949	65.46	11.88	0.1133^{-1}
8.00	0.1024^{-3}	0.1414^{-2}	0.7246^{-1}	190.1	0.3929	74.50	13.39	0.8488^{-2}
8.50	0.6898^{-4}	0.1066^{-3}	0.6472^{-1}	251.1	0.3912	84.13	14.99	0.6449^{-2}
9.00	0.4739^{-4}	0.8150^{-3}	0.5814^{-1}	327.2	0.3898	94.33	16.69	0.4964^{-2}
9.50	0.3314^{-4}	0.6313^{-3}	0.5249^{-1}	421.1	0.3886	105.1	18.49	0.3866^{-2}
10.00	0.2356^{-4}	0.4948^{-3}	0.4762^{-1}	535.9	0.3876	116.5	20.39	0.3045^{-2}

$^*x^{-n}$ means $x \cdot 10^{-n}$

Compressible Flow - Table

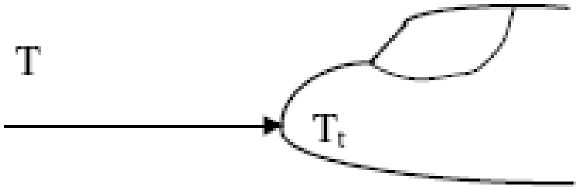


Properties of the U.S. Standard Atmosphere (SI Units)

Altitude (m)	Temperature (C°)	Acceleration of Gravity, g (m/s^2)	Pressure, p [Pa, abs]	Density, ρ (kg/m^3)	Dynamic Viscosity, μ (Pa.s)
- 1,000	21.50	9.810	1.139E+5	1.347E+0	1.821E - 5
0	15.00	9.807	1.013E+5	1.225E+0	1.789E - 5
1,000	8.50	9.804	8.988E+4	1.112E+0	1.758E - 5
2,000	2.00	9.801	7.950E+4	1.007E+0	1.726E - 5
3,000	- 4.49	9.797	7.012E+4	9.093E- 1	1.694E - 5
4,000	-10.98	9.794	6.166E+4	8.194E- 1	1.661E - 5
5,000	-17.47	9.791	5.405E+4	7.364E- 1	1.628E - 5
6,000	-23.96	9.788	4.722E+4	6.601E- 1	1.595E - 5
7,000	- 30.45	9.785	4.111E+4	5.900E- 1	1.561E - 5
8,000	- 36.94	9.782	3.565E+4	5.258E- 1	1.527E - 5
9,000	- 43.42	9.779	3.080E+4	4.671E- 1	1.493E - 5
10,000	- 49.90	9.776	2.650E+4	4.135E- 1	1.458E - 5
15,000	-56.50	9.761	1.211E+4	1.948E- 1	1. 422E - 5

Class 18: Compressible Flow

Problem: **For an aircraft flying at Mach 3.0 at an altitude of 10,000 m (T = -50°C), estimate the surface temperature at the nose.**



Solution: **From the table, @ M = 3,**

$$\frac{T}{T_t} = 0.3571$$

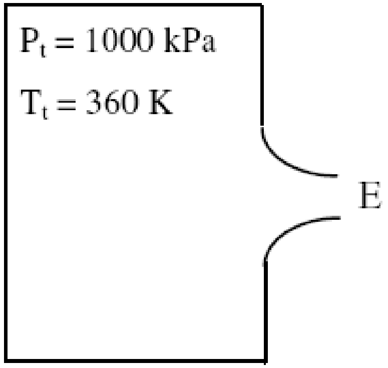
$$\Rightarrow \frac{T_t}{T} = 2.80$$

$$\Rightarrow T_t = 2.8\left(273-50\right) = 624^o K = 351^o C$$

Class 18: Compressible Flow

Problem: **A converging nozzle has an exit area of 500mm^2. Air enters this nozzle from a reservoir at 1000kpa & 360K. The exit pressure is 800kPa. Find the mass flow rate through the nozzle.**

Solution: **First we determine the pressure ratio,**



$$\frac{P_E}{P_t} = \frac{800}{1000} = 0.8(subsonic > 0.5283)$$

Using this value,we get from the table

$$M_E = 0.573$$

$$\Rightarrow \frac{T_E}{T_t} = 0.9381$$

$$\Rightarrow T_E = 0.9381 \times 360 = 337.7^o K$$

Now

$$c_E = \sqrt{kRT_E} = \sqrt{(1.4)(287)(337.7)} = 368.4\, m/s$$

Hence

$$V_E = M_E \cdot c_E = (0.573)\cdot(368.4\, m/s) = 211.1 m/s$$

$$\rho_E = \frac{P_E}{RT_E} = \frac{800 \times 10^3\ N/m^2}{(287\, J/kg.^o K)(337.7^o K)} = 8.254\, kg/m^3$$

$$\dot{m} = \rho_E V_E A_E = (8.254\, kg/m^3)(211.1 m/s)(500 \times 10^{-6} m^2)$$

$$\Rightarrow \dot{m} = 0.871 kg/s$$

Class 18: Compressible Flow

Problem: **The stagnation pressure indicated by a Pitot tube mounted on an airplane in flight is 45 kPa (abs). If the aircraft is cruising in standard atmosphere at an altitude of 10,000m, determine the speed and Mach number involved.**

Solution: **First we find the pressure ratio**

$$P = 2.65\times10^4\,Pa,\quad P_t = 45\times10^3\,Pa$$

$$\Rightarrow \frac{P}{P_t} = 0.589\ (\text{subsonic} > 0.5283)$$

Now from the table, we get

$$M \cong 0.90$$

Therefore,

$$V = M\cdot c = M\sqrt{kRT}$$

$$\Rightarrow V = 0.90\sqrt{(1.4)(287)(273-49.90)} = 269\,m/s$$