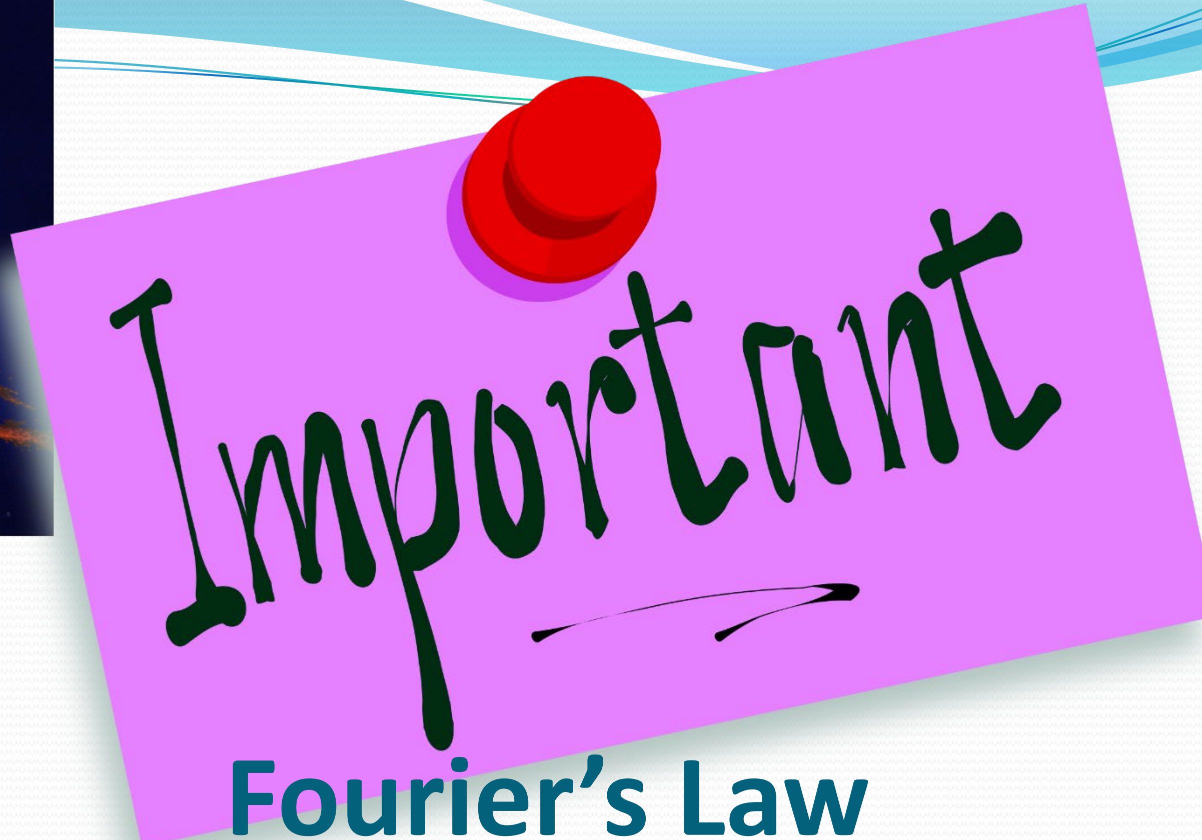


Fourier's Law and the Heat Diffusion Equation

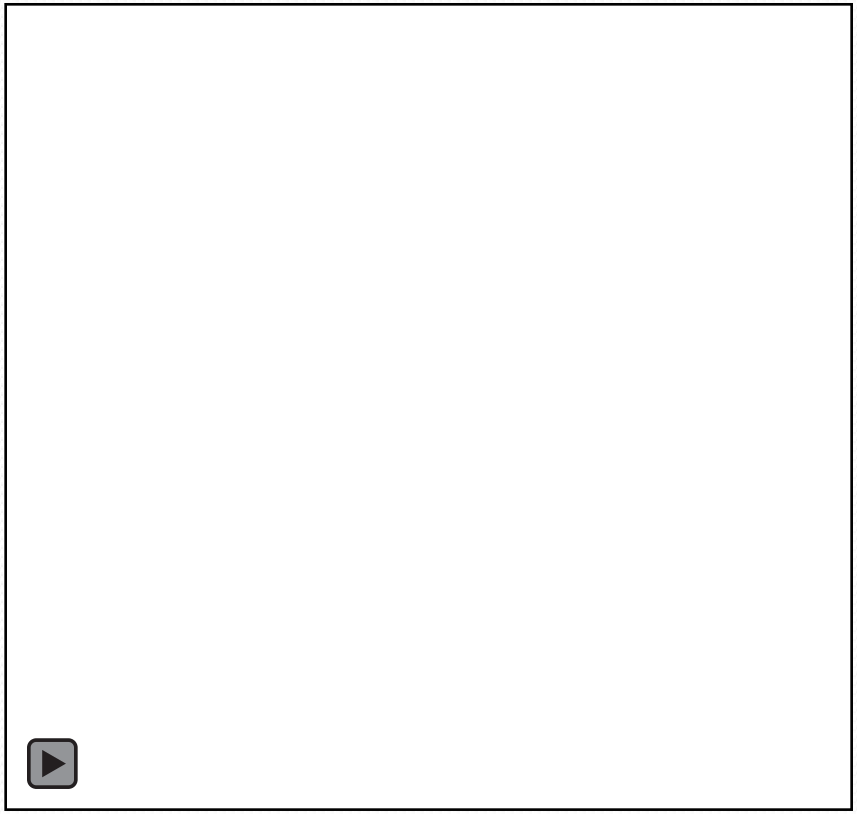
- **Chapter Two**

ZOOM CLASS ROOM


TRUST THE PATH
Seek Wisdom Do You? Do, or do not, there is no try.

ENGINEERING
PRODUCT DEVELOPMENT

- Developing engineering solutions to improve mankind using applied math for idea creation and analysis, product development, and realization.



FLUIDS STUDENT *WINTER 2021*

What suggestion would you provide to future students to enhance their understanding and performance within ME-322/420 Fluid Mechanics/Heat Transfer?

Follow that path. Every problem in this class has a similar thought process for understanding what is happening. It is also very important to have a very strong understanding of the basic aspects of the fluid dynamics, the first 3-5 weeks will teach you the most important aspects that will be critical to solving problems for the rest of this class and for problems in your future. You must understand the basics before you dive into the deep end of the pool.

What advice would you provide to MECH-322/420 Fluid/Heat Transfer students in Dr. Berry's class to enhance their success and performance?

Make sure to complete the homework but find a way to make sure your answers are correct, practice done incorrectly will instill bad habits when trying to solve problems. Find a problem, make sure you can do it correctly, then make sure you can do it over and over correctly each time. If you can understand the fundamentals, every other problem is the same path you only start from a different location.

NO SUCCESS WITHOUT PRACTICE

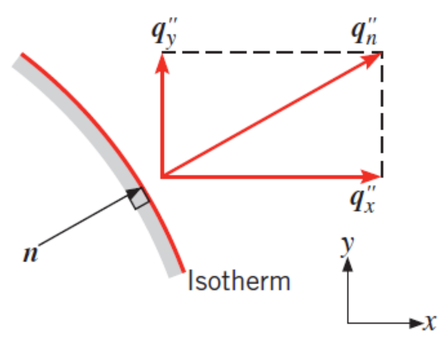
Fourier's Law

$$q'' = -k\nabla T$$

- A ***RATE EQUATION*** that allows determination of the conduction heat flux from knowledge of the temperature distribution in a medium
- Its most general (vector) form for multidimensional conduction is:

Implications:

– Heat transfer is in the direction of decreasing temperature (basis for minus sign).



– Fourier's Law serves to define the thermal conductivity of the medium

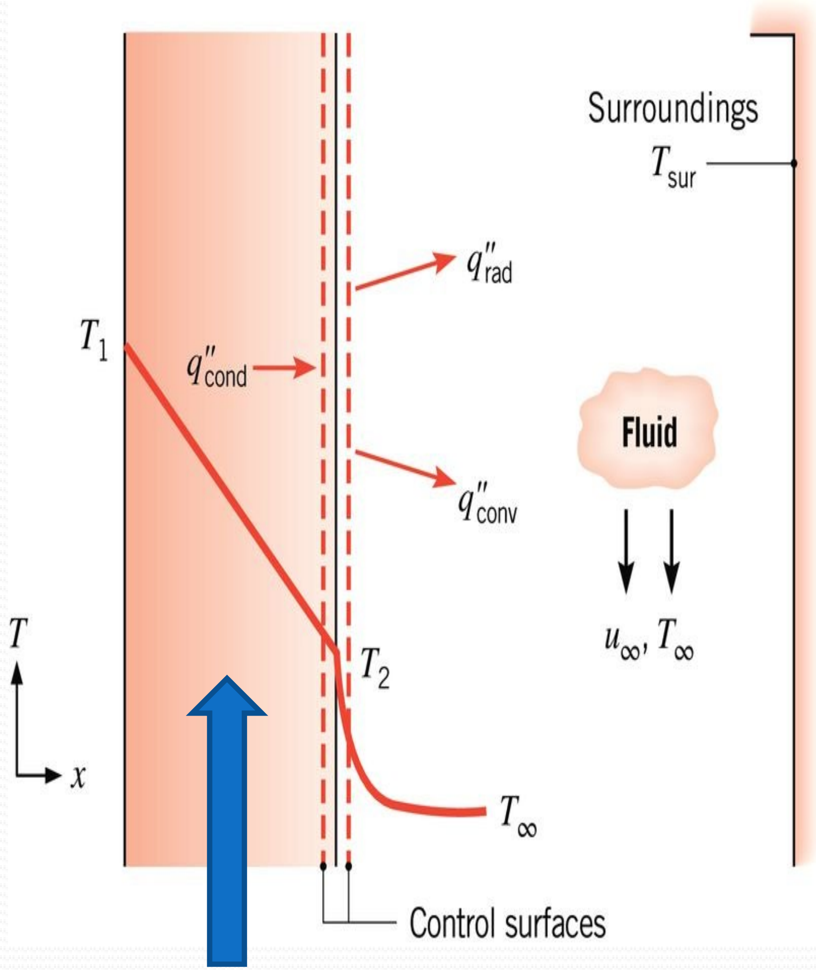
– Direction of heat transfer is perpendicular to ***lines of constant temperature*** (isotherms).

– Heat flux vector may be resolved into orthogonal components.

"When" is the CV 1st Law Needed

- <u>To find SURFACE temperature only, Ts. (and heat flux is known at each surface)</u>
  - Apply 1st law to CV around entire object
- To find initial rate of change of temperature (dT/dt)
- When there is no spatial gradients of temperature "INSIDE" the body, i.e. a "thin walled tube".
- Apply at "surface" to find energy balance at surface to determine boundary conditions.

Apply 1st Law to SURFACE



$$\overset{+}{\dot{E}}_{in} - \overset{+}{\dot{E}}_{out} + \cancel{\overset{\pm}{\dot{E}}_{g}} = \frac{dE_{st}}{dt} \equiv \cancel{\overset{\pm}{\dot{E}}_{st}}$$

$$\overset{+}{\dot{E}}_{in} - \overset{+}{\dot{E}}_{out} = 0$$

$$q''_{conduction} - (q''_{convection} + q''_{radiation}) = 0$$

ONLY CONCERNED ABOUT SURFACE TEMPERATURE.

KNOW FLUX AT ALL BOUNDARIES

BUT!!!! HOW DO WE FIND TEMP INSIDE OF MEDIUM, I.E. T(X) = ?

HEAT DIFFUSION EQUATION

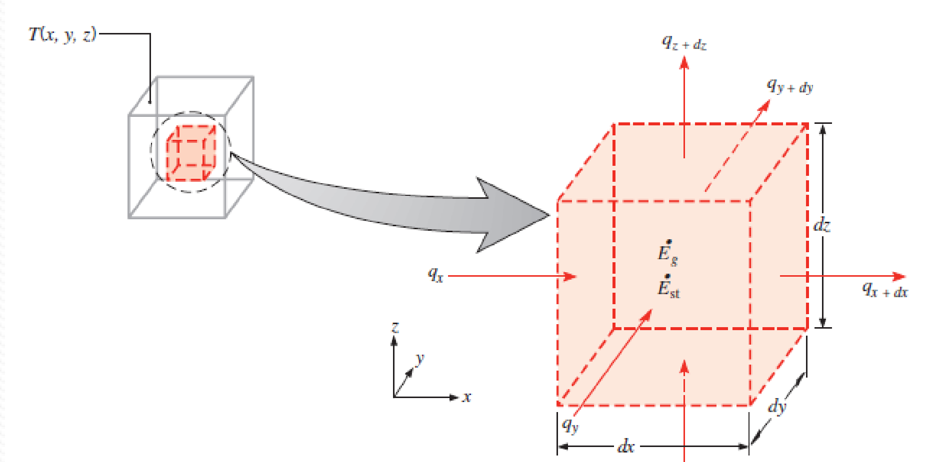
Heat Diffusion Equation ??

- 2^{nd} Order PDE used to find temperature "INSIDE" body everywhere. i.e T(x, y,z,time).
    - *Use Fourier's Law to now find q(x,y,z,time).*
- Solve PDE to find (T(x) inside, and even find surface temperature, Ts = T(x=L).
- Requires information at boundary to find arbitrary constants of integration. Need boundary conditions.
    - Use 1^{st} law applied to each "surface" to determine *heat flux* boundary conditions.

The Heat Equation

- A differential equation whose solution provides the temperature distribution in a stationary medium.
- Based on applying conservation of energy to a differential control volume through which energy transfer is exclusively by conduction.
- Cartesian Coordinates:

DERIVE One-D and Two-D HDE



3D Heat Diffusion Equation

$$\underbrace{\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)}+\dot{S}_{gen}(x,y,z,t)=\rho c_p\frac{\partial T}{\partial t} \qquad (2.19)$$

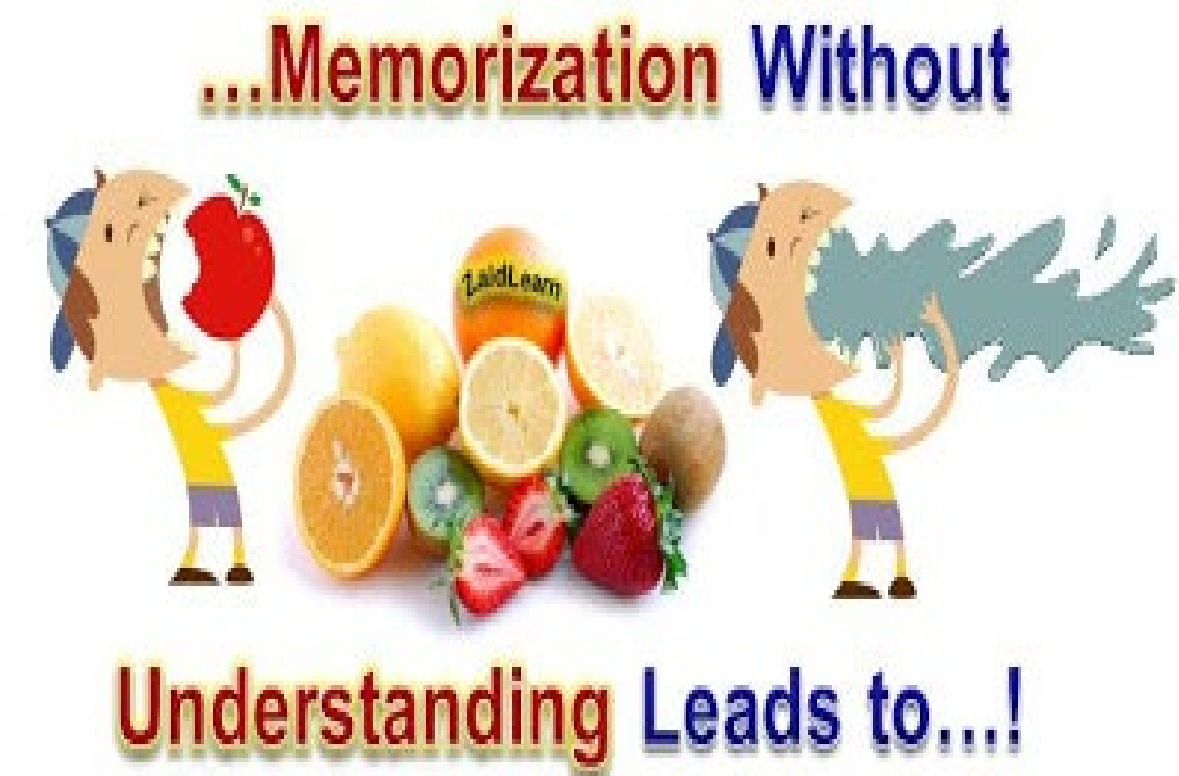
Net transfer of thermal energy into the control volume (inflow-outflow)

Thermal energy generation

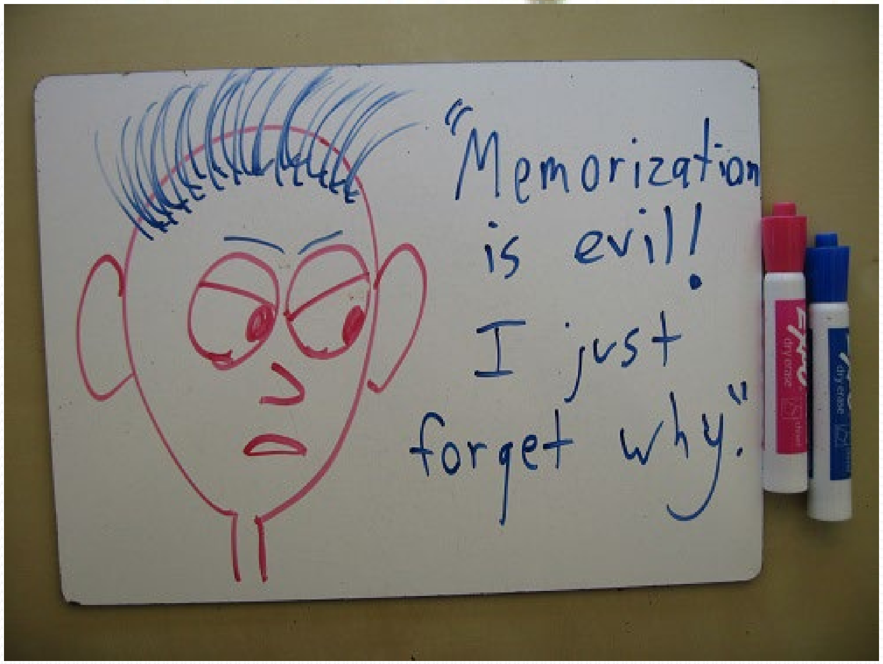
Change in thermal energy storage

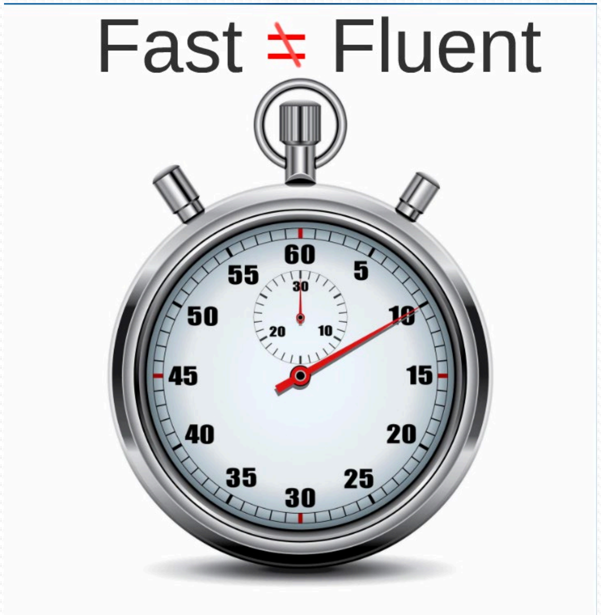
MECH-422 Heat Transfer 1D HDE CARTESIAN STUDY AID

Dr. K. J. Berry
ASME FELLOW

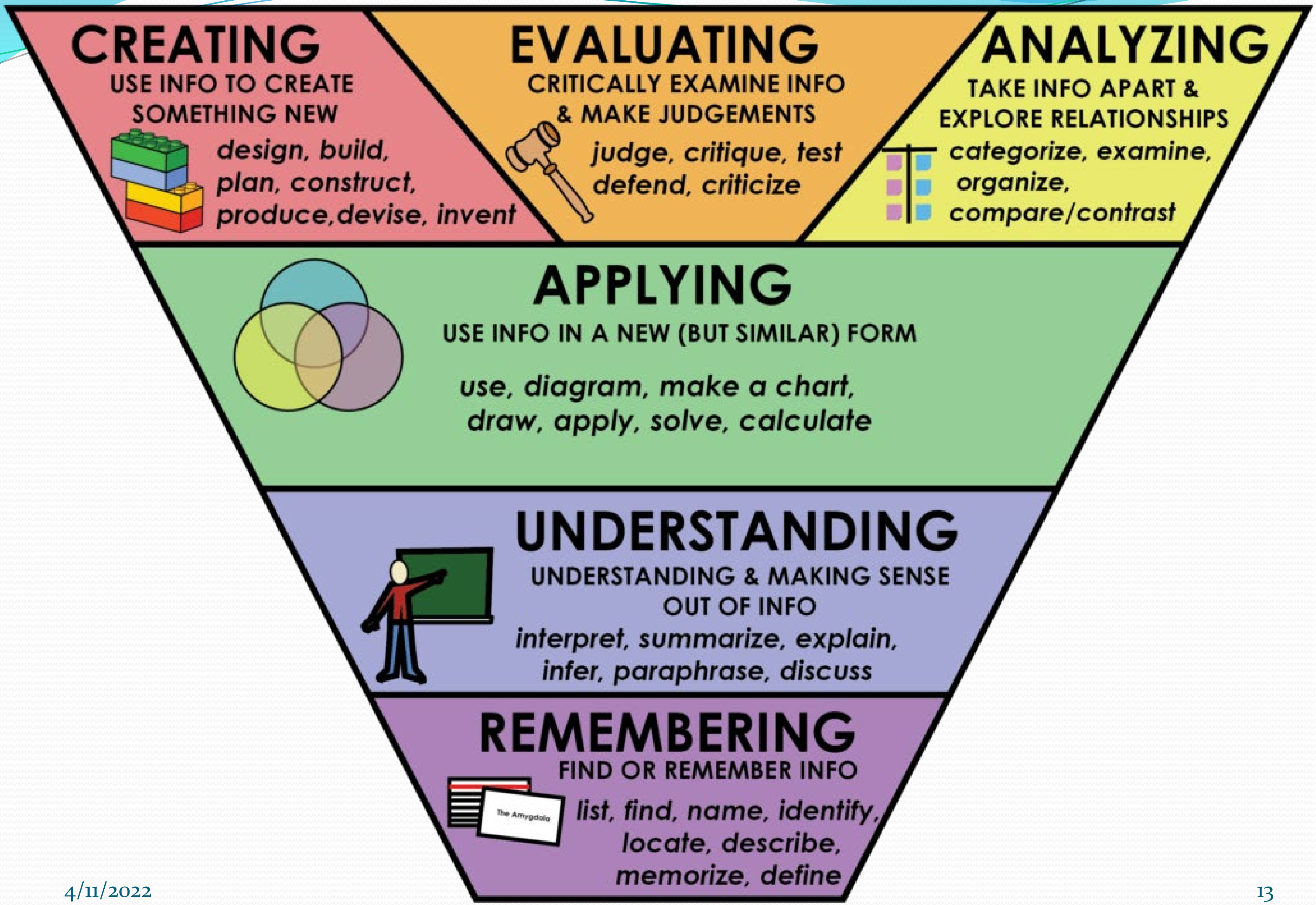

...Memorization Without
ZaidLearn
Understanding Leads to...!


CHAOS


"Memorization is evil! I just forget why."


Fast ≠ Fluent

BLOOM'S TAXONOMY



CORE UNDERSTANDING

To find <u>T(x,y,z,t)</u> INSIDE medium, we MUST solve 2nd ORDER PDE with Boundary Conditions to obtain EXACT SOLUTIONS.

NO OPTIONS

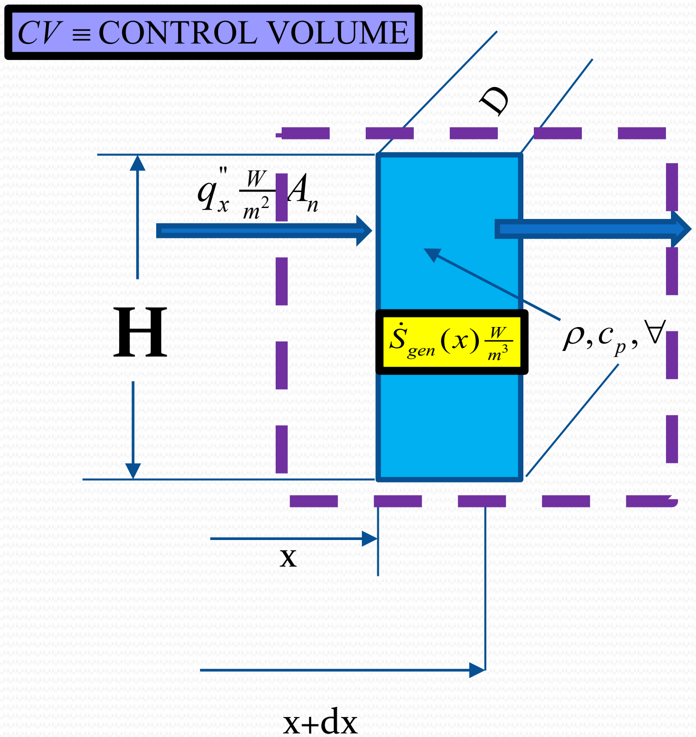
NO EXCEPTIONS

NO EXCUSES

1D HDE Cartesian

$d\forall \equiv$ Differentia Volume $\equiv H \bullet D \bullet dx$

$A_n \equiv$ Normal Area to Heat Transfer $\equiv H \bullet D \neq F(x)$



$$q_{x+dx}'' \tfrac{W}{m^2} A_n = q_x'' \tfrac{W}{m^2} A_n + \frac{d}{dx}\left[q_x'' \tfrac{W}{m^2} A_n\right]dx$$

1st Law Applied to Control Volume

$$\dot{E}_{in} - \dot{E}_{out} \pm \dot{E}_{gen} = \dot{E}_{st} = \frac{d}{dt}(E_{st})$$

$$\dot{E}_{in} = q_x'' \tfrac{W}{m^2} A_n$$

$-$

$$\dot{E}_{out} = q_{x+dx}'' \tfrac{W}{m^2} A_n = q_x'' \tfrac{W}{m^2} A_n + \frac{d}{dx}\left[q_x'' \tfrac{W}{m^2} A_n\right]dx$$

$+$

$$\dot{E}_{gen} = \dot{S}_{gen}(x) \tfrac{W}{m^3} d\forall$$

$=$

$$E_{st} = \rho c_p d\forall \frac{dT}{dt}$$

$$-\frac{d}{dx}\left[q_x'' \tfrac{W}{m^2}\right] \cancel{HDdx} + \dot{S}_{gen}(x) \tfrac{W}{m^3} \cancel{HDdx} = \rho c_p \cancel{HDdx} \frac{dT}{dt}$$

1D HDE **CARTESIAN** FINAL FORM

$$-\frac{d}{dx}\left[q_x^{"}\,\frac{W}{m^2}\right]\cancel{HDdx}+\dot{S}_{gen}(x)\frac{W}{m^3}\cancel{HDdx}=\rho c_p\cancel{HDdx}\frac{dT}{dt}$$

HDE

$$-\frac{d}{dx}\left[q_x^{"}\,\frac{W}{m^2}\right]+\dot{S}_{gen}(x)\frac{W}{m^3}=\rho c_p\frac{dT}{dt}$$

FOURIER's LAW

$$q_x^{"}=-k_x\frac{dT}{dx}$$

$$-\frac{d}{dx}\left[-k_x\frac{dT}{dx}\frac{W}{m^2}\right]+\dot{S}_{gen}(x)\frac{W}{m^3}=\rho c_p\frac{dT}{dt}$$

REDUCTIONS

HOMOGENOUS: k_x=constant $\neq$ f(x)

$$\frac{d^2T}{dx^2}+\frac{\dot{S}_{gen}(x)}{\mathrm{k}_x}=\frac{\rho c_p}{\mathrm{k}_x}\frac{dT}{dt}=\frac{1}{\alpha}\frac{dT}{dt}$$

$\alpha \equiv$ *THERMAL* DIFFUSIVITY$=\dfrac{k_x}{\rho c_p}\left[\dfrac{m^2}{s}\right]$;measure of speed of heat diffusion

MOST GENERAL SOLUTION

- Assume **steady state 1D** Heat Transfer in a plain **wall** of width L, height H and depth D. Determine the most "**GENERAL SOLUTION** ". Assume **homogeneous** medium for temperature T(x) within medium.
- Start with general form of HDE:

$$\frac{d^2T}{dx^2}+\frac{\dot{S}_{gen}(x)}{k_x}=\frac{\rho c_p}{k_x}\frac{dT}{dt}=\frac{1}{\alpha}\frac{\cancel{dT}}{\cancel{dt}}=0;\ \text{Steady State}$$

- Integrate with respect to "x" to express in terms of arbitrary constants of integration.

$$\frac{d^2T}{dx^2}+\frac{\dot{S}_{gen}(x)}{k_x}=0$$

$$\frac{d^2T}{dx^2}=-\frac{\dot{S}_{gen}(x)}{k_x}$$

$$\frac{dT}{dx}=\int_x -\frac{\dot{S}_{gen}(x)}{k_x}dx+C_1$$

$$T(x)=\int_x\int_x\left[-\frac{\dot{S}_{gen}(x)}{k_x}dx\right]dx+C_1\bullet x+C_2;\ 0\le x\le L$$

Most General Solution until we specify:

1. BOUNDARY CONDITIONS

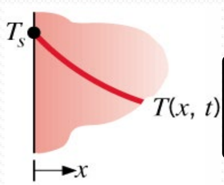
2. Form of internal heat generation rate, $\dot{S}_{gen}(x)$

EXACT SOLUTION ROADMAP

- NEED BOUNDARY CONDITIONS (one condition at each boundary):

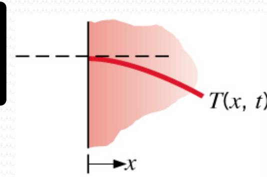
2nd Order in Space Requires Two (2) Boundary Conditions
1st Order in Time requires One (1) Initial Condition

Constant Surface Temperature:



Boundary Condition
$\left(T(x = X^*, t\right) = T_s \rightarrow$ CONST. TEMP.

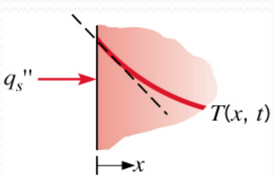
Insulated Surface



Boundary Condition
$\left(\frac{dT}{dx}\right)_{x=X^*} = 0 \rightarrow INSULATION$

Constant Heat Flux:

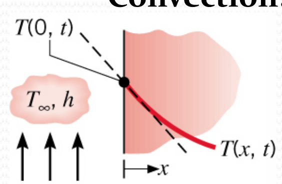
Applied Flux



Boundary Condition
$\left(-k\frac{dT}{dx}\right)_{x=X^*} = q_s'' \rightarrow$ HEAT FLUX

Convection:

$-k\frac{\partial T}{\partial x}\Big|_{x=0} = h\left[T_\infty - T(0,t)\right]$



Boundary Condition
$\left(-k\frac{dT}{dx}\right)_{x=X^*} = h(T_s - T_\infty) \rightarrow CONVECTION$

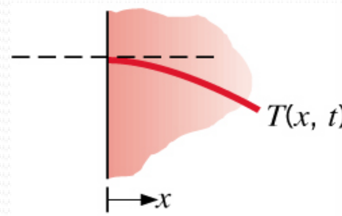
Find *PARAMETRIC* ROAD MAP *EXACT* Solution

A plain wall experiences an internal heat generation rate of the form:

$$\dot{S}_{gen}(x) = S_0 \frac{W}{m^3} \sin(\frac{\pi x}{L}); 0 \le x \le L$$

with Boundary Conditions defined as:

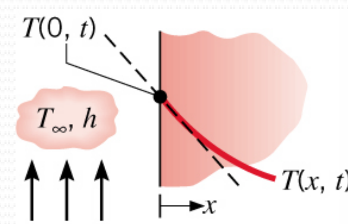
1) x=0; Insulated, $\rightarrow \frac{dT}{dx}_{x=0} = 0$



2) $x = L$, Convective Fluid.

$\rightarrow q''_{conduction} = q''_{convection}$

$\rightarrow -k_x \frac{dT}{dx}_{x=L} = h(T_s - T_\infty) = h(T_{(x=L)} - T_\infty)$



$$T(x) = \int_x \int_x \left[-\frac{S_0 \frac{W}{m^3} \sin(\frac{\pi x}{L})}{k_x} dx \right] dx + C_1 \bullet x + C_2; 0 \le x \le L$$

$$T(x) = \frac{S_0}{k_x} \left(\frac{L}{\pi} \right)^2 \sin(\frac{\pi x}{L}) + C_1 \bullet x + C_2$$

FIND C1 and C2

$$T(x)=\int_x\int_x\left[-\frac{S_0\frac{W}{m^3}\sin(\frac{\pi x}{L})}{k_x}dx\right]dx+C_1\bullet x+C_2;\ 0\le x\le L$$

$$T(x)=\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)^2\sin(\frac{\pi x}{L})+C_1\bullet x+C_2$$

$$\frac{dT}{dx}=\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)^1\cos(\frac{\pi x}{L})+C_1$$

$BC\#1:$

$$\frac{dT}{dx}_{x=0}=0=\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)^1\cos(\frac{\pi\bullet 0}{L})+C_1$$

$$C_1=-\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\rightarrow\frac{W/m^3}{W/m-K}m\rightarrow\frac{K}{m};\ \text{UNIT CHECK}$$

$$T(x)=\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)^2\sin(\frac{\pi x}{L})-\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\bullet x+C_2$$

$$T(x)=\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)^2\sin(\frac{\pi x}{L})-\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\bullet x+C_2$$

$BC\#2:$

$$-k_x\frac{dT}{dx}_{x=L}=h(T(x=L)-T_\infty)$$

$$-k_x\left[\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\cos(\frac{\pi L}{L})-\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\right]=h\left[-\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\bullet L+C_2-T_\infty\right]$$

$$C_2=\frac{-k_x\left[\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\cos(\frac{\pi L}{L})-\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\right]}{h}+\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\bullet L+T_\infty$$

UNIT CHECK

$$\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\bullet L\rightarrow\frac{\frac{W}{m^3}}{\frac{W}{m-K}}m^2\rightarrow K$$

$$\frac{S_0}{h}\left(\frac{L}{\pi}\right)\rightarrow\frac{\frac{W}{m^3}}{\frac{W}{m^2-K}}m\rightarrow K$$

$$C_2=\frac{\left[-S_0\left(\frac{L}{\pi}\right)(1+\cos(\frac{\pi L}{L})\right]}{h}+\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\bullet L+T_\infty$$

$$C_2=\frac{2S_0L}{\pi h}+\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\bullet L+T_\infty$$

EXACT SOLUTION

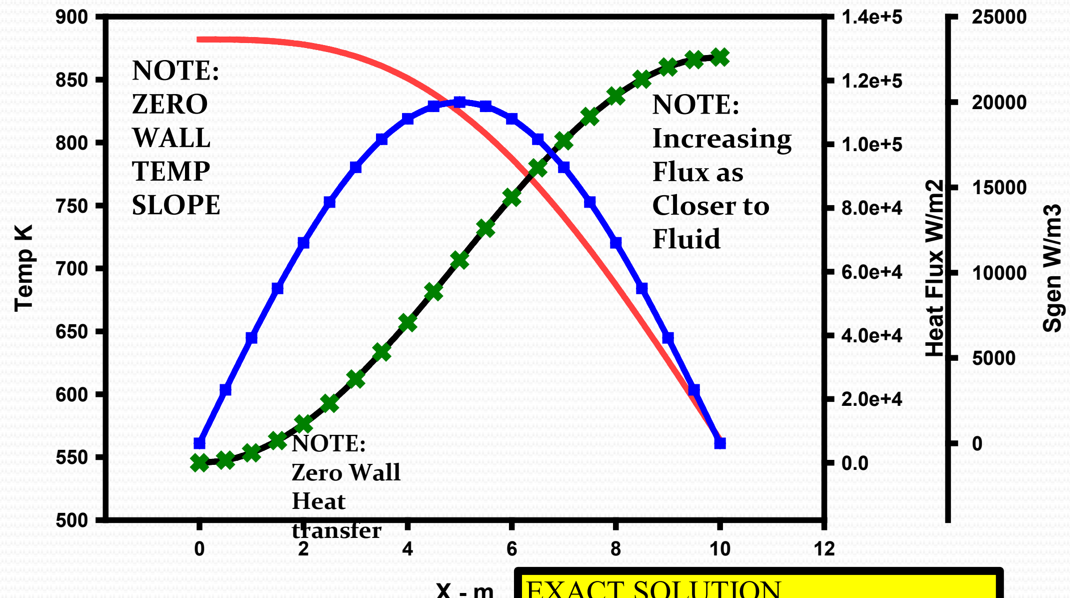
$$T(x)=\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)^2\sin(\frac{\pi x}{L})-\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\bullet x+C_2$$

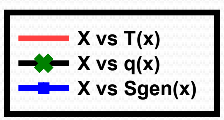
$$C_2=\frac{2S_0L}{\pi h}+\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\bullet L+T_\infty$$

W/m3	m	W/m-K	W/m2-K	K	K/m	K
S0	L	Kx	h	Tf	C1	C2
20000	10	2000	2000	500	-31.83102	881.9721861

X	T(x)	q(x)	S(x)
0	881.9722	0	0
0.5	881.9068	783.7839	3128.687
1	881.4512	3115.836	6180.335
1.5	880.2246	6938.735	9079.803
2	877.8653	12158.35	11755.7
2.5	874.0396	18646.15	14142.13
3	868.4498	26242.39	16180.33
3.5	860.8416	34760.02	17820.12
4	851.0104	43989.32	19021.12
4.5	838.8065	53703.02	19753.76
5	824.1385	63661.95	20000
5.5	806.9755	73620.88	19753.77
6	787.3485	83334.58	19021.14
6.5	765.3487	92563.89	17820.15
7	741.1259	101081.5	16180.36
7.5	714.8847	108677.8	14142.16
8	686.8794	115165.6	11755.74
8.5	657.4077	120385.3	9079.85
9	626.8033	124208.2	6180.385
9.5	595.4279	126540.3	3128.739
10	563.6623	127324.1	0.053072

1D HDE Solution
Wall Sinusoidal Heat Generation





EXACT SOLUTION

$$T(x) = \frac{S_0}{k_x}\left(\frac{L}{\pi}\right)^2 \sin\left(\frac{\pi x}{L}\right) - \frac{S_0}{k_x}\left(\frac{L}{\pi}\right) \bullet x + C_2$$

$$C_2 = \frac{2S_0 L}{\pi h} + \frac{S_0}{k_x}\left(\frac{L}{\pi}\right) \bullet L + T_\infty$$

Find Surface Temperature Two (2) Methods

EXACT SOLUTION

$$T(x)=\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)^2\sin(\frac{\pi x}{L})-\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\bullet x+C_2$$

$$C_2=\frac{2S_0L}{\pi h}+\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\bullet L+T_\infty$$

$$T_s=T(x=L)=\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)^2 \mathrm{si} \ldots$$

$$=-\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\bullet L+C_2$$

$$=-\frac{S_0}{k_x}\left(\frac{L}{\pi}\right)\bullet L+881.97\ldots$$

$$=-\frac{20,000W/m^3}{2000W/m-K}\frac{100m^2}{\pi}+8\ldots$$

$$=563.7K$$

OVERALL ENERGY BALANCE

$$-\dot{E}_{out}+\dot{E}_{gen}=0$$

$$\dot{E}_{gen}=\int S_{gen}(x)d\forall=HD\int_V S_{gen}(x)dx$$

$$\ldots x)dx=-HDS_0\frac{L}{\pi}\cos(\frac{\pi x}{L})_{0-L}=-HDS_0\frac{L}{\pi}(-1-1)$$

$$=20,000W/m^3\frac{20m}{\pi 2000W/m^2-K}+500K$$

$$=563.66K$$

KNOW YOUR HEAT DIFFUSION EQUATION (HDE) (SPECIAL CASE)

- One-Dimensional Conduction in a Planar Medium with Constant Properties and **NO GENERATION**

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) = \rho c_p \frac{\partial T}{\partial t}$$

becomes

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\alpha \equiv \frac{k_x}{\rho c_p} \rightarrow \text{ thermal diffusivity of the medium } \left[\mathrm{m}^2/\mathrm{s}\right]$$

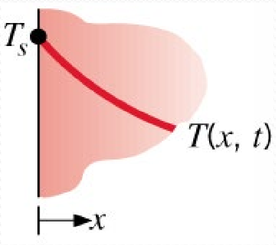
Boundary and Initial Conditions

- For transient conduction, Heat Diffusion Equation (HDE) is first order in time, requiring specification of an initial temperature distribution:
- Since ***HEAT DIFFUSION EQUATION (HDE)*** is second order in space, two boundary conditions, must be specified. Some common cases:

2nd Order in Space Requires Two (2) Boundary Conditions

1st Order in Time requires One (1) Initial Condition

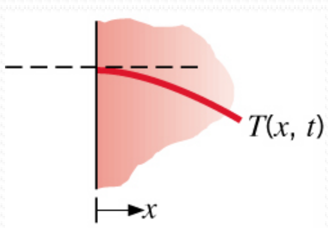
Constant Surface Temperature:



Boundary Condition

$$\left(T(x=X^*,t)\right)=T_s \rightarrow \text{CONST. TEMP.}$$

Insulated Surface

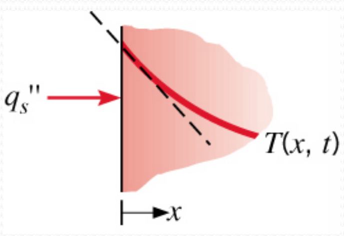


Boundary Condition

$$\left(\frac{dT}{dx}\right)_{x=X^*}=0\rightarrow INSULATION$$

Constant Heat Flux:

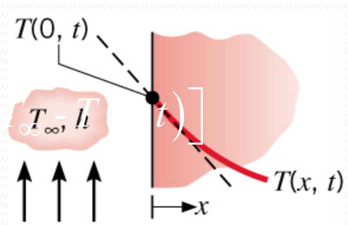
Applied Flux



Boundary Condition

$$\left(-k\frac{dT}{dx}\right)_{x=X^*}=q_s'' \rightarrow \text{HEAT FLUX}$$

Convection:

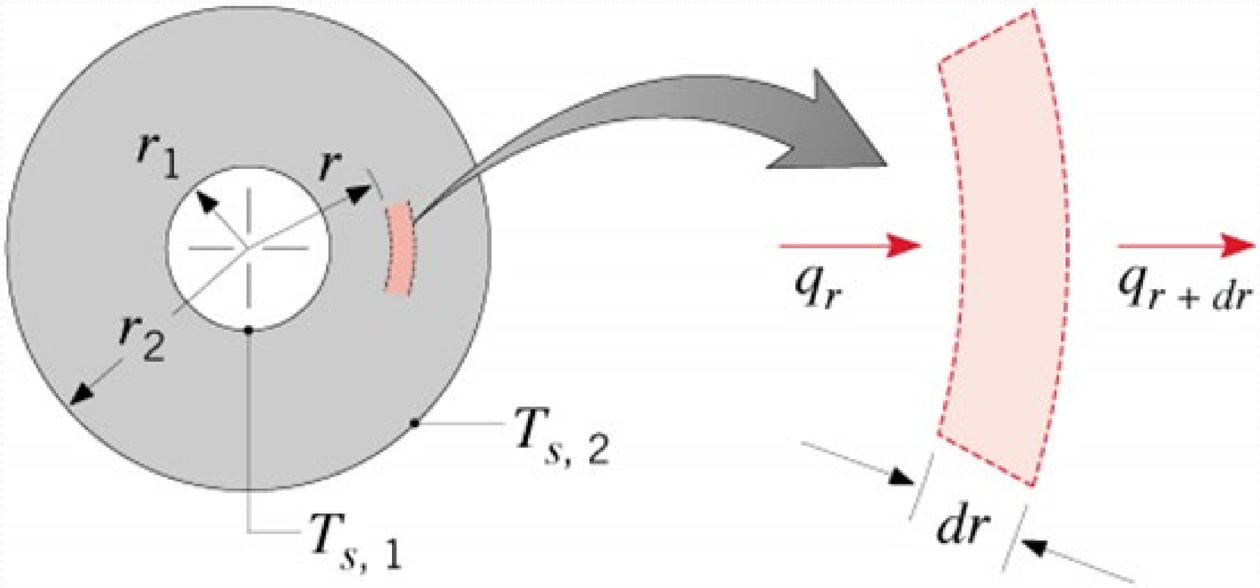


Boundary Condition

$$\left(-k\frac{dT}{dx}\right)_{x=X^*}=h(T_s-T_\infty)\rightarrow CONVECTION$$

HDE + BC Solutions--**CYLINDRICAL**

$$q_r = A_r q_r'' = 2\pi r L q_r'' \; Watts$$

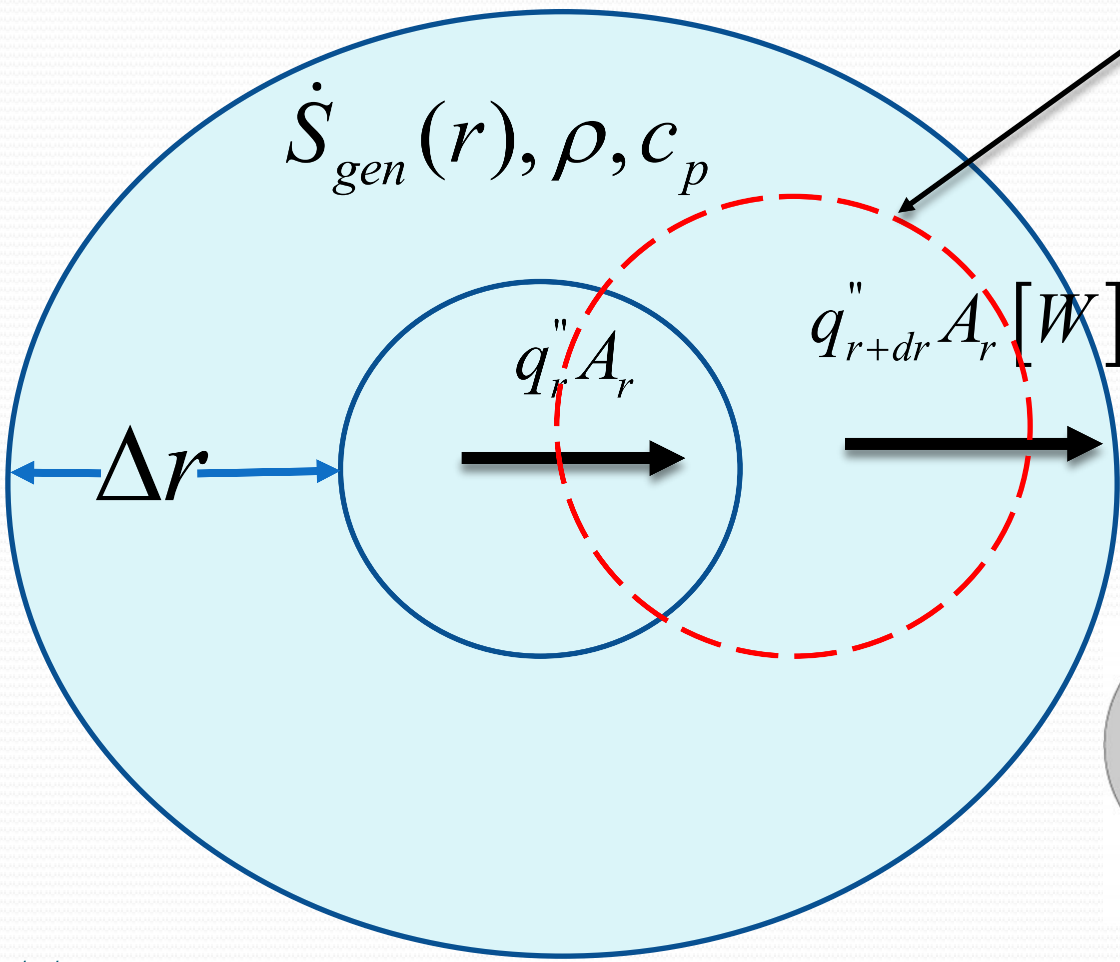


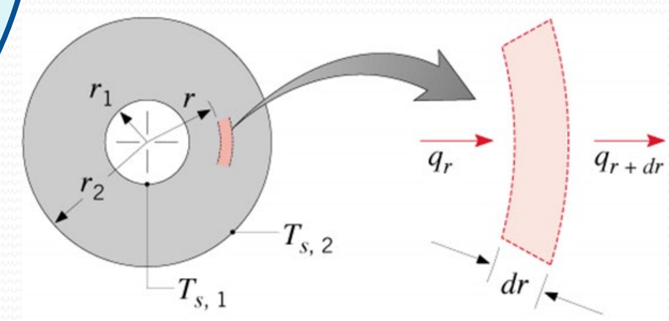
1D Cylindrical HDE

$A_r \equiv$ Radial Area Normal to Heat Transfer:2πrL

CONTROL VOLUME

$$q''_{r+dr}A_r\left[W\right] = q''_r A_r + \frac{d}{dr}\left[q''_r A_r\right]dr$$





1D Cylindrical HDE
Apply 1st Law

$$\overset{+}{\dot{E}}_{\text{in}} - \overset{+}{\dot{E}}_{\text{out}} \pm \overset{+}{\dot{E}}_{gen} = \frac{dE_{\text{st}}}{dt} \equiv \rho \forall c_p \frac{dT}{dt}$$

$$\overset{+}{\dot{E}}_{\text{in}} = q''_r A''_r$$

$$\overset{+}{\dot{E}}_{\text{out}} = q''_r A_r + \frac{d}{dr}\left[q''_r A_r \right] dr$$

$$\overset{+}{\dot{E}}_{\text{in}} - \overset{+}{\dot{E}}_{out} = -\frac{d}{dr}\left[q''_r A_r \right] dr$$

$$A_r = 2\pi r \bullet L \text{ (normal to Heat Transfer)}$$

$$\overset{+}{\dot{E}}_{gen} = \dot{S}_{gen}(r) \frac{W}{m^3} \bullet d\forall = \dot{S}_{gen}(r) \frac{W}{m^3} \bullet A''_r \bullet dr$$

1D Cylindrical HDE

$$\overset{+}{\dot{E}}_{\text{in}} - \overset{+}{\dot{E}}_{\text{out}} \pm \overset{+}{\dot{E}}_{gen} = \frac{dE_{\text{st}}}{dt} \equiv \rho \forall c_p \frac{dT}{dt}$$

$$-\frac{d}{dr}(q_r'' 2\pi rL)dr + \dot{S}_{gen}(r) \bullet 2\pi rL \bullet dr = \rho 2\pi rL \bullet dr \bullet c_p \frac{dT}{dt}$$

$$-\frac{d}{dr}(q_r'' r)\cancel{2\pi L}dr + \dot{S}_{gen}(r) \bullet \cancel{2\pi L}dr \bullet r = \rho \cancel{2\pi L}dr \bullet r \bullet c_p \frac{dT}{dt}$$

$$-\frac{1}{r}\frac{d}{dr}(q_r'' r) + \dot{S}_{gen}(r) = \rho \bullet c_p \frac{dT}{dt}$$

Fourier's Law

$$q_r'' = -k_r \frac{dT}{dr}$$

$HDE \rightarrow$

$$\frac{1}{r}\frac{d}{dr}(k_r r \frac{dT}{dr}) + \dot{S}_{gen}(r) = \rho \bullet c_p \frac{dT}{dt}$$

1D Cylindrical HDE

$$\frac{1}{r}\frac{d}{dr}(k_r r\frac{dT}{dr}) + \dot{S}_{gen}(r,t) = \rho \bullet c_p \frac{dT}{dt}$$

REDUCTIONS

Homogeneous $\rightarrow \mathrm{k}_r \neq F(r)$

$$\frac{1}{r}\frac{d}{dr}(r\frac{dT}{dr}) + \frac{\dot{S}_{gen}(r,t)}{k_r} = \frac{\rho \bullet c_p}{k_r}\frac{dT}{dt} = \frac{1}{\alpha}\frac{dT}{dt}$$

STEADY STATE $\rightarrow \frac{dT}{dt} = 0$

$$\frac{1}{r}\frac{d}{dr}(r\frac{dT}{dr}) + \frac{\dot{S}_{gen}(r)}{k_r} = 0$$

Heat Flux Components (cont.)

- In angular coordinates $(\phi \text{ or } \phi, \theta)$, the temperature gradient is still based on temperature change over a length scale and hence has units of °C/m and not °C/deg.

- Heat rate for one-dimensional, radial conduction in a cylinder or sphere:
  - Cylinder

$$q_r = A_r q_r'' = \left(2\pi r L\left[m^2\right]\right) q_r''\left[\frac{W}{m^2}\right] \rightarrow Watts$$

or,

$$q_r' = A_r' q_r'' = \left(2\pi r[m]\right) q_r''\left[\frac{W}{m^2}\right] \rightarrow \frac{Watts}{Length}$$

  - Sphere

$$q_r = A_r q_r'' = \left(4\pi r^2\left[m^2\right]\right) q_r''\left[\frac{W}{m^2}\right] \rightarrow Watts$$

HDE SUMMARY

2nd Order PDE Used to find temperature "INSIDE" everywhere, T(x,y,z,t)

Use Fourier's Law to now find $q(x, y, z, time)$.

1D Cartesian

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \dot{S}_{gen}(x,t) = \rho c_p \frac{\partial T}{\partial t}$$

2D Cartesian

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + \dot{S}_{gen}(x,y,t) = \rho c_p \frac{\partial T}{\partial t}$$

1D Spherical

$$\frac{1}{r^2}\frac{d}{dr}(k_r r^2 \frac{dT}{dr}) + \dot{S}_{gen}(r,t) = \rho c_p \frac{dT}{dt}$$

1D Cylindrical

$$\frac{1}{r}\frac{d}{dr}(k_r r \frac{dT}{dr}) + \dot{S}_{gen}(r,t) = \rho c_p \frac{dT}{dt}$$

2D Cylindrical

$$\frac{1}{r}\frac{d}{dr}(k_r r \frac{dT}{dr}) + \frac{d}{dz}(k_z \frac{dT}{dr}) + \dot{S}_{gen}(r,z,t) = \rho c_p \frac{dT}{dt}$$

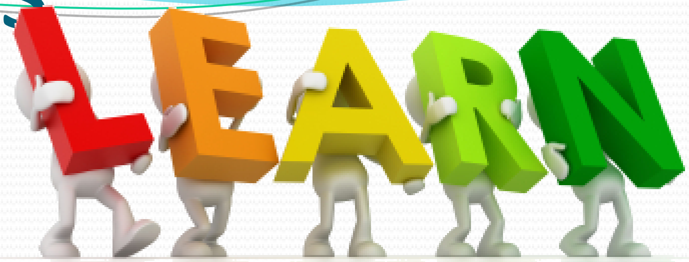
$$\alpha \equiv \frac{k}{\rho c_p} \rightarrow$$ thermal diffusivity of the medium $\left[\mathrm{m}^2/\mathrm{s}\right]$

1D Exact Solutions w/BC's (QUIZ)

- SOLVE
  - Temp-Temp
  - Temp-Insulation
  - Temp-Convection
  - Temp-Heat Flux

2nd Order in Space Requires Two (2) Boundary Conditions
1st Order in Time requires One (1) Initial Condition

Learning Expectations



1. Solve 1D HDE for Cartesian, Cylindrical and Spherical to obtain EXACT solution for T(x), T(r); and variable Sgen(x), Sgen (r).

2. Be able to obtain solution for any of the three specified cases for Boundary Conditions

Typical Methodology of a Conduction Analysis

- Consider possible microscale or nanoscale effects in problems involving very small physical dimensions or very rapid changes in heat or cooling rates.

- Solve appropriate form of heat equation to obtain the temperature distribution.

- Knowing the temperature distribution, apply Fourier's Law to obtain the heat flux at any time, location and direction of interest.

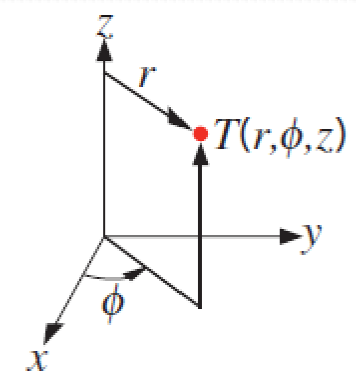
- Applications:

  Chapter 3: One-Dimensional, Steady-State Conduction
  Chapter 4: Two-Dimensional, Steady-State Conduction
  Chapter 5: Transient Conduction

Heat Flux Components

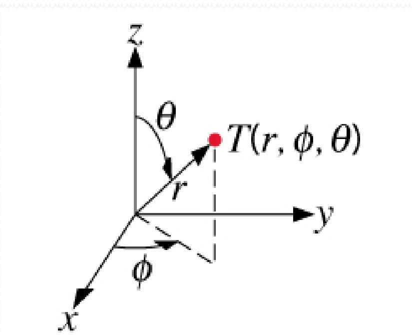
- Cartesian Coordinates:

$$\vec{q}'' = \left(\underbrace{-k_x \frac{\partial T}{\partial x}}_{}\vec{i}\right) - \left(\underbrace{k_y \frac{\partial T}{\partial y}}_{}\vec{j}\right) - \left(\underbrace{k_z \frac{\partial T}{\partial z}}_{}\vec{k}\right) \qquad (2.3)$$



- Cylindrical Coordinates:

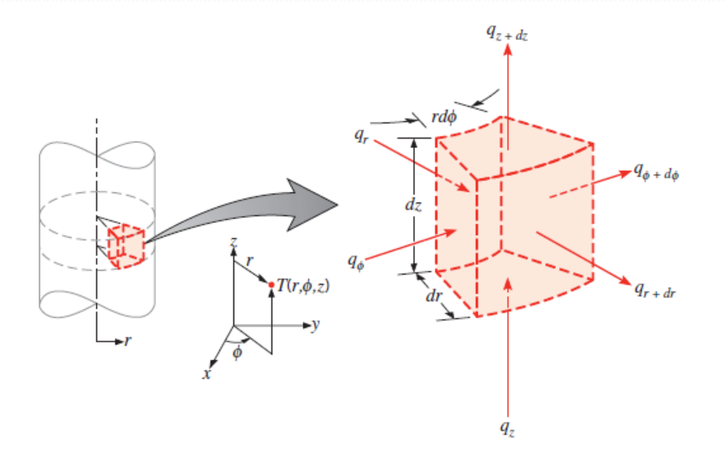
$$\vec{q}'' = \underbrace{-k_r \frac{\partial T}{\partial r}}_{}\vec{i} - \left(\underbrace{k_\phi \frac{\partial T}{r\partial \phi}}_{}\vec{j}\right) - \left(\underbrace{k_z \frac{\partial T}{\partial z}}_{}\vec{k}\right) \qquad (2.24)$$



- Spherical Coordinates:

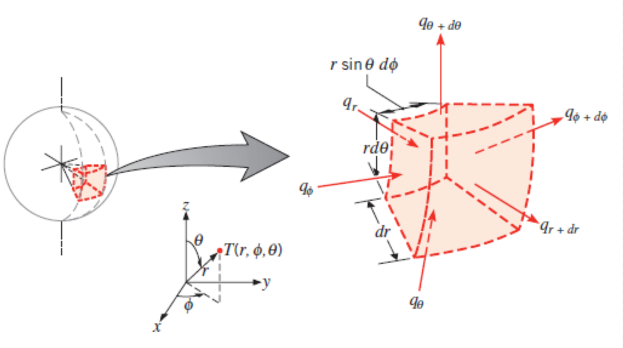
$$\vec{q}'' = \underbrace{-k_r \frac{\partial T}{\partial r}}_{}\vec{i} - \left(\underbrace{k_\theta \frac{\partial T}{r\partial \theta}}_{}\vec{j}\right) - \left(\underbrace{k_\phi \frac{\partial T}{r\sin\theta\,\partial \phi}}_{}\vec{k}\right) \qquad (2.27)$$

- Cylindrical Coordinates:



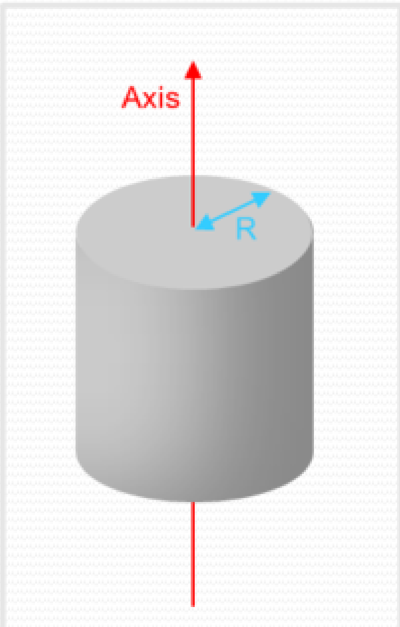
$$\frac{1}{r}\frac{\partial}{\partial r}\left(k_r r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k_\phi\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)+\dot{S}_{gen}(x,y,z,t)=\rho c_p\frac{\partial T}{\partial t} \qquad (2.26)$$

- Spherical Coordinates:



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(k_r r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k_\phi\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k_\theta\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{S}_{gen}(x,y,z,t)=\rho c_p\frac{\partial T}{\partial t} \qquad (2.29)$$

Derive **GENERAL SOLUTION** for 1D CYLINDER



$$\frac{1}{r}\frac{d}{dr}(k_r r\frac{dT}{dr}) + \dot{S}_{gen}(r,t) = \rho c_p \frac{dT}{dt}$$

Steady State/Homegeneous

$$\frac{1}{r}\frac{d}{dr}(r\frac{dT}{dr}) = \frac{-\dot{S}_{gen}(r)}{k_r}; \quad 0 \le r \le r_0$$

Multiple by r

$$\frac{d}{dr}(r\frac{dT}{dr}) = \frac{-\dot{S}_{gen}(r)\bullet r}{k_r}$$

Integrate Once

$$r\frac{dT}{dr} = \int\left(\frac{-\dot{S}_{gen}(r)\bullet r}{k_r}\right)dr + C_1$$

÷ by r

$$\frac{dT}{dr} = \left(\frac{1}{r}\int\left(\frac{-\dot{S}_{gen}(r)\bullet r}{k_r}\right)dr\right) + \frac{C_1}{r}$$

INTEGRATE AGAIN

$$T(r) = \int\left(\frac{1}{r}\int\left(\frac{-\dot{S}_{gen}(r)\bullet r}{k_r}\right)dr\right)dr + C_1\ln(r) + C_2$$

Case 1: Sgen = 0, Solid Cylinder

$T(r) = C_1 \ln(r) + C_2$

EXACT SOLUTION:

HEAT FLUX

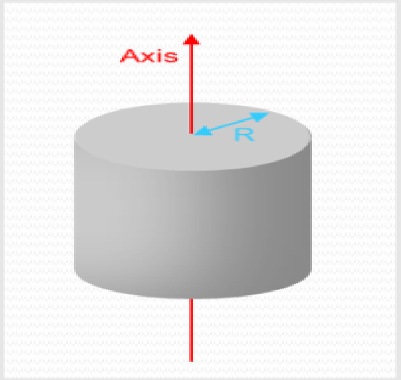
$$q''_r \frac{W}{m^2} = -k_r \frac{dT}{dr} = -k_r \frac{C_1}{r} = 0$$

B.C's:

#1 $@r=0 \rightarrow but\ \ln(0) \rightarrow -\infty$

#2 $@r=r_0$

#1: $T(r=0) \rightarrow$ Must Exist & Be Finite $\rightarrow C_1 = 0$



$T(r) = C_2$

NOTE: T(r) is not a function of $k_r \rightarrow$ HOW POSSIBLE ?

NOTE : T(r) is a constant for solid cylinder and SS, and Sgen = 0.

FINAL: $T(r) = T_0$

Case 2: Sgen=0, ***Cylindrical SHELL***

$$\mathrm{T(r)=C_1}\ln(r)+C_2; \quad r_1 \le r \le r_2$$

EXACT SOLUTION:

$$BC\#1: T(r=r_1)=T_1, BC\#2: T(T=r_2)=T_2$$

$$\#1: T_1 = \mathrm{C_1}\ln(r_1)+C_2, C_2 = T_1\text{-}\mathrm{C_1}\ln(r_1)$$

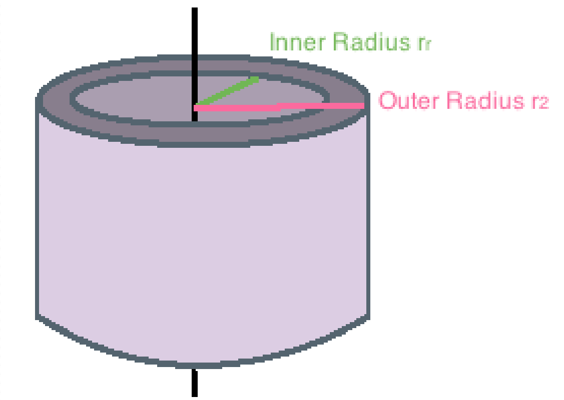
$$\#2: T_2 = \mathrm{C_1}\ln(r_2)+C_2$$

$$T_2 = \mathrm{C_1}\ln(r_2)+T_1\text{-}\mathrm{C_1}\ln(r_1)$$

$$C_1 = \frac{T_2-T_1}{\ln(r_2)-\ln(r_1)} = \frac{T_2-T_1}{\ln(r_2/r_1)} \to [\frac{K}{\ln(r)}]$$

$$C_2 = T_1\text{-}\mathrm{C_1}\ln(r_1)$$

$$C_2 = T_1\text{-}\frac{T_2-T_1}{\ln(r_2/r_1)}\ln(r_1) \to [K]$$



Final, Sgen=0, Steady, k=Const

$$T(r) = C_1 \ln(r) + C_2$$

$$\text{T(r)}= \frac{T_2 - T_1}{\ln(r_2 / r_1)} \ln(r) + T_1 - \frac{T_2 - T_1}{\ln(r_2 / r_1)} \ln(r_1)$$

$$\text{T(r)}= T_1 + \frac{T_2 - T_1}{\ln(r_2 / r_1)} \left(\ln \frac{r}{r_1} \right); \quad r_1 \le r \le r_2$$

HEAT FLUX

$$\text{q}''_r \frac{W}{m^2} = -k_r \frac{dT}{dr} = -k_r \frac{C_1}{r} = -k_r \frac{\frac{T_2 - T_1}{\ln(r_2 / r_1)}}{r}$$

$$q(r)[W] = \text{q}''_r \frac{W}{m^2} A_r = \text{q}''_r \frac{W}{m^2} \bullet 2\pi r L$$

HDE SPHERE – SS Exact Solution

- **SOLID SPHERE**

$HOMEGENOUS$ – –SOLID SPHERE

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dT}{dr}) + \frac{\cancel{\dot{S}_{gen}(r,t)}}{k_r} = \cancel{\frac{\rho c_p}{k_r}\frac{dT}{dt}} = 0 \rightarrow \text{INTEGRATE TWICE}$$

$$T(r) = -\frac{C_1}{r} + C_2; 0 \le r \le r_0 \rightarrow \text{MOST GENERAL SOLUTION}$$

BOUNDARY CONDITIONS - SPECIAL CASE A

1. $\mathrm{T(r{=}0)} \rightarrow \mathrm{FINITE} \rightarrow \mathrm{C_1} = 0$
2. $\mathrm{T(r{=}r_0)} = T_s(\mathrm{known}) \rightarrow \mathrm{C_2} = T_s$

$EXACT$

$$T(r) = T_s$$

$$q_r'' = -k_r\frac{dT}{dr} = 0$$

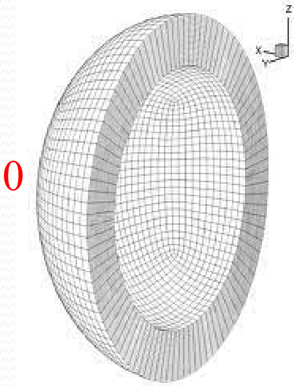
OTHER BOUNDARY CONDITIONS WILL RESULT IN OTHER EXACT SOLUTIONS.

HDE SPHERE – SS Exact Solution

- **SPHERICAL SHELL**

HOMEGENOUS – –SOLID SPHERE

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)+\frac{\cancel{\dot{S}_{gen}(r,t)}}{k_r}=\cancel{\frac{\rho c_p}{k_r}\frac{dT}{dt}}=0$$



$$T(r)=-\frac{C_1}{r}+C_2;\, r_1\le r\le r_2 \rightarrow$$

MOST GENERAL SOLUTION

BOUNDARY CONDITIONS - SPECIAL CASE A

1. $T(r=r_1)=T_1$
2. $T(r=r_2)=T_2$

EXACT

$$1.\, T(r=r_1)=T_1=-\frac{C_1}{r_1}+C_2\rightarrow C_2=T_1+\frac{C_1}{r_1}\rightarrow[K]$$

$$2.\, T(r=r_2)=T_2=-\frac{C_1}{r_2}+T_1+\frac{C_1}{r_1}=T_1+C_1\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

$$T_2=T_1+C_1\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\rightarrow C_1=\frac{T_2-T_1}{\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}\rightarrow[K\bullet m]$$

EXACT SOLUTION

$$1.\, C_2=T_1+\frac{C_1}{r_1}$$

$$2.\, C_1=\frac{T_2-T_1}{\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}\rightarrow C_2=T_1+\frac{1}{r_1}\frac{T_2-T_1}{\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}$$

$$T(r)=-\frac{1}{r}C_1+C_2$$

$$T(r)=-\frac{1}{r}\frac{T_2-T_1}{\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}+T_1+\frac{1}{r_1}\frac{T_2-T_1}{\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}$$

$$T(r)=T_1+\frac{T_2-T_1}{\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}\left(\frac{1}{r_1}-\frac{1}{r}\right);\, r_1\le r\le r_2$$

OTHER BOUNDARY CONDITIONS WILL RESULT IN OTHER EXACT SOLUTIONS.

HEAT FLUX – EVERYWHERE

Spherical Shell

$$T(r) = T_1 + \frac{T_2 - T_1}{(\frac{1}{r_1} - \frac{1}{r_2})}(\frac{1}{r_1} - \frac{1}{r}); r_1 \le r \le r_2$$

$$q_r''(r) = -k\frac{dT}{dr} = -k\left[\frac{C_1}{r^2}\right]; C_1 = \frac{T_2 - T_1}{(\frac{1}{r_1} - \frac{1}{r_2})}$$

$$q_r''(r) = -k\frac{dT}{dr} = -\frac{k}{r^2}\frac{T_2 - T_1}{(\frac{1}{r_1} - \frac{1}{r_2})}$$

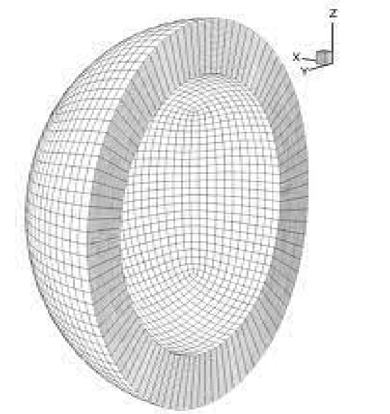
$$\rightarrow \left[\frac{\frac{W}{m-K}}{m^2}\right][K - m] \rightarrow \frac{W}{m^2}$$

$$q_r''(r) \sim r^2$$

$$q[W] = q_r''(r) \bullet A_{r\,sphere} = q_r''(r) \bullet 4\pi r^2$$

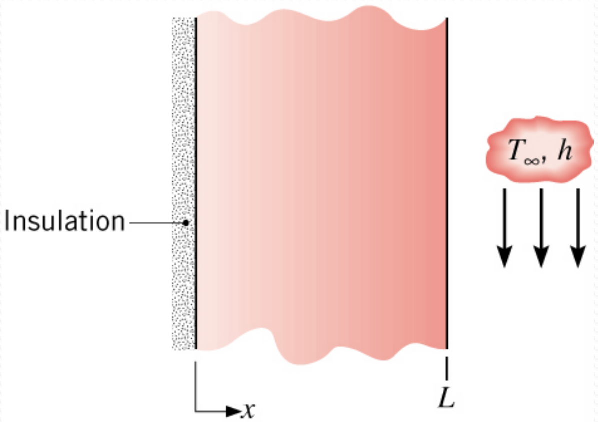
$$q[W] = -\frac{k}{r^2}\frac{T_2 - T_1}{(\frac{1}{r_1} - \frac{1}{r_2})} \bullet 4\pi r^2$$

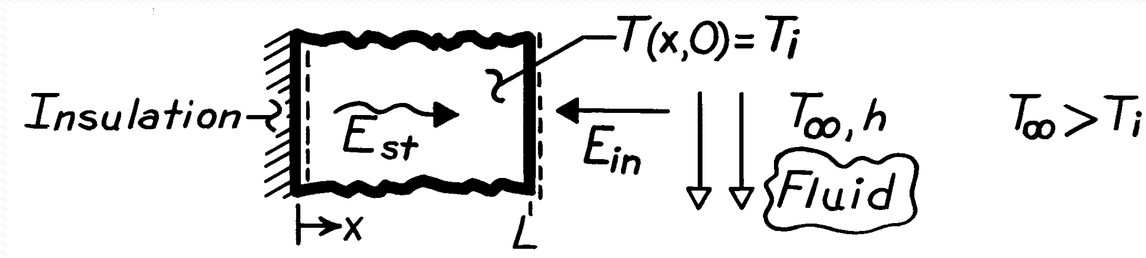
$$q[W] = -4\pi k\frac{T_2 - T_1}{(\frac{1}{r_1} - \frac{1}{r_2})} \rightarrow \text{CONSTANT} \ne \text{F(r)}$$



OTHER BOUNDARY *CONDITIONS WILL RESULT IN OTHER EXACT* SOLUTIONS.

Thermal Response to Plane Wall





Known: Plane Wall, initially at uniform temperature, is suddenly exposed to convective heating.
Find: Differential equation, boundary and initial condition to find T(x,t)?

HDE

$$\frac{\partial^2 T}{\partial x^2} = \frac{\rho c_p}{k}\frac{\partial T}{\partial t} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Initial Condition

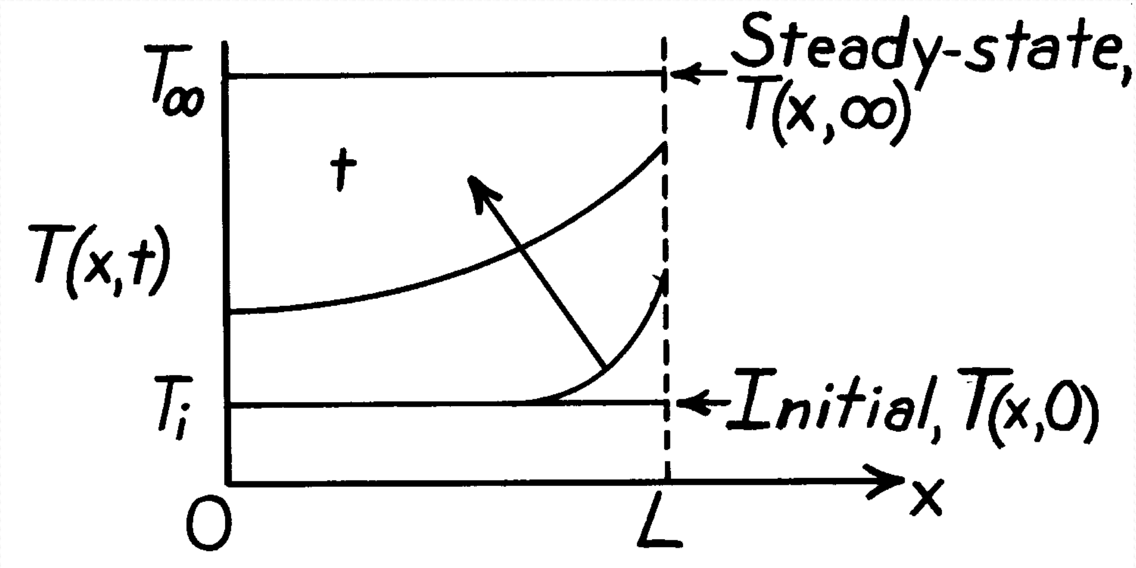
$T(x,t=0)=T_i$

Boundary Conditions

$$1.\ \frac{dT}{dx}_{x=0} = 0, \quad 2. -k\frac{dT}{dx}_{x=L} = h(T_\infty - T(x=L))$$

Thermal Response to Plane Wall

Sketch “INITIAL”, “STEADY STATE” and 2 intermediate temperature profiles.



$$T_\infty > T_i$$

Note that the gradient at x=0 is always "0" due to wall insulation.
Note that the gradient at x=L, decreases with as we approach SS.

TOTAL ENERGY TO WALL (W)

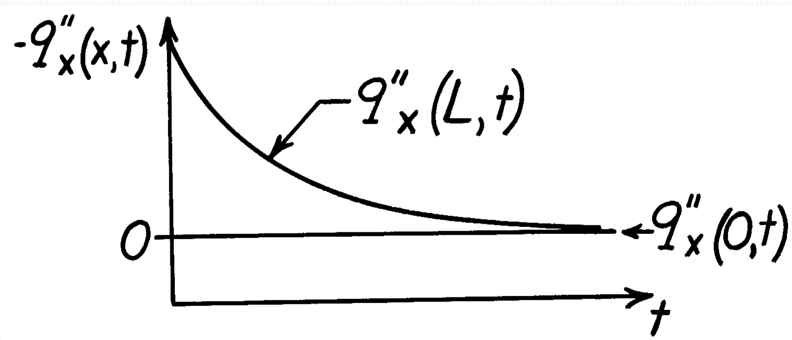
$$E_{in} = \int_{0}^{\infty} q_s^{"}(x = L, t)\frac{W}{m^2}A_s dt$$

$$E_{in} = hA_s \int_0^{\infty} \left(T_{\infty} - T\left(L,t\right)\right) dt \rightarrow [J]$$

TOTAL ENERGY TO WALL (W/V)

$$\frac{E_{in}}{V} = \frac{hA_s}{V = A_s L}\int_0^{\infty} \left(T_{\infty} - T\left(L,t\right)\right) dt \rightarrow \left[\frac{J}{m^3}\right]$$

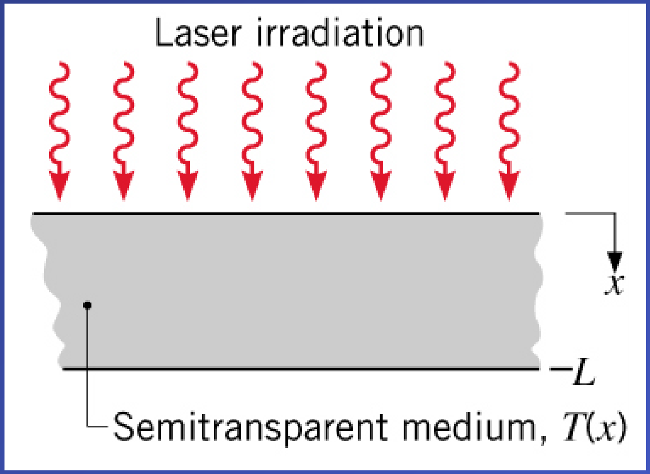
Heat Flux vs Time (Wall and Surface)



NOTE: For all times, heat flux at wall (x=0) does not change, since wall boundary condition at x=0 does not change.

NOTE: As times becomes large at steady state, the surface heat flux (x=L), becomes ZERO, as heat is transferred from the convective fluid to the entire WALL temperature **everywhere** will eventually approach that of the fluid with **NO INTERNAL HEAT GENERATION.**

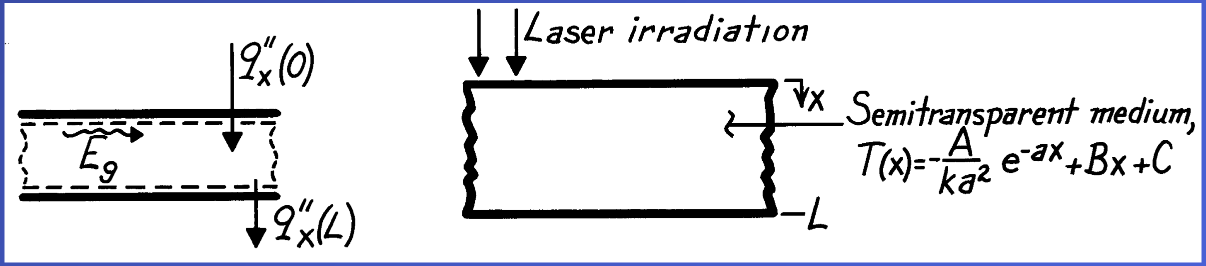
Problem 2.37 Surface heat fluxes, heat generation and total rate of radiation absorption in an irradiated semi-transparent material with a prescribed temperature distribution.



KNOWN: Temperature distribution in a semi-transparent medium subjected to radiative flux.

FIND: (a) Expressions for the heat flux at the front and rear surfaces, (b) The heat generation rate $\dot{q}(x)$, and (c) Expression for absorbed radiation per unit surface area.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in medium, (3) Constant properties, (4) All laser irradiation is absorbed and can be characterized by an internal volumetric heat generation term $\dot{q}(x)$.

ANALYSIS: (a) Knowing the temperature distribution, the surface heat fluxes are found using Fourier's law,

$$q''_x = -k\left[\frac{dT}{dx}\right] = -k\left[-\frac{A}{ka^2}(-a)e^{-ax} + B\right]$$

$$\boxed{T(x) = -\frac{A}{ka^2}e^{-ax} + Bx + C}$$

Front Surface, x=0: $$q''_x(0) = -k\left[+\frac{A}{ka}\bullet 1 + B\right] = -\left[\frac{A}{a} + kB\right]$$ <

Rear Surface, x=L: $$q''_x(L) = -k\left[+\frac{A}{ka}e^{-aL} + B\right] = -\left[\frac{A}{a}e^{-aL} + kB\right]$$ <

(b) The heat diffusion equation for the medium is

$$\frac{d}{dx}\left(\frac{dT}{dx}\right) + \frac{\dot{q}}{k} = 0 \quad or \quad \dot{q} = -k\frac{d}{dx}\left(\frac{dT}{dx}\right)$$

$$\dot{q}(x) = -k\frac{d}{dx}\left[+\frac{A}{ka}e^{-ax} + B\right] = Ae^{-ax}.$$

$$\boxed{\dot{q} \equiv \dot{S}_{gen}(x,y,z,t) \rightarrow \frac{W}{m^3}}$$

(c) Performing an energy balance on the medium,

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = 0$$

On a unit area basis

$$\dot{E}''_g = -\dot{E}''_{\text{in}} + \dot{E}''_{\text{out}} = -q''_x(0) + q''_x(L) = +\frac{A}{a}\left(1 - e^{-aL}\right). \qquad <$$

Alternatively, evaluate $\dot{E}''_g$ by integration over the volume of the medium,

$$\dot{E}''_g = \int_0^L \dot{q}(x)dx = \int_0^L Ae^{-ax}dx = -\frac{A}{a}\left[e^{-ax}\right]_0^L = \frac{A}{a}\left(1 - e^{-aL}\right).$$

NOTE: ON A UNIT AREA BASIS (WIDTH x DEPTH) -> NORMAL TO HEAT FLOW

FOR A PLAIN WALL

$$\dot{E}_{gen}[W] = A_n \int_0^L \dot{S}_{gen}(x)\left[\frac{W}{m^3}\right]dx, A_n \rightarrow \text{AREA NORMAL TO HEAT FLOW}$$

$$\dot{E}''_{gen}\left[\frac{W}{m^2}\right] = \frac{\dot{E}_{gen}[W]}{A_n} = \int_0^L \dot{S}_{gen}(x)\left[\frac{W}{m^3}\right]dx$$

SYSTEMS STUDY

On the military outpost Natilas Prime, a new tera forming device has
a bio radiation signature of the form:

$$\dot{S}_{gen}(x) = S_0(1 - e^{-\beta x})\left[\frac{W}{m^3}\right], S_0, \beta \rightarrow \text{Constants}$$

To contain the radiation, the homegeneous unit (k_B=50W/m-K) is placed within a rectangular containment vessel with an adiabatic wall (x=0), and at the other side (x=L) there is a lead radiatio insulation barrier of thickness "t" that experiences a convective heat transfer fluid due to the strong gale force winds 800 of MPH with a convective heat transfer coefficienct of h=450 W/m^2-K during the darkest nights at a temperature of T_∞=2C.

As chief thermal systems engineer you are requested to study "lead" thickness and impact on SS interface temperature and external surface temperature, as function of S_0 and β.


TRUST THE PATH
Seek Wisdom Do You? Do, or do not, there is no try.

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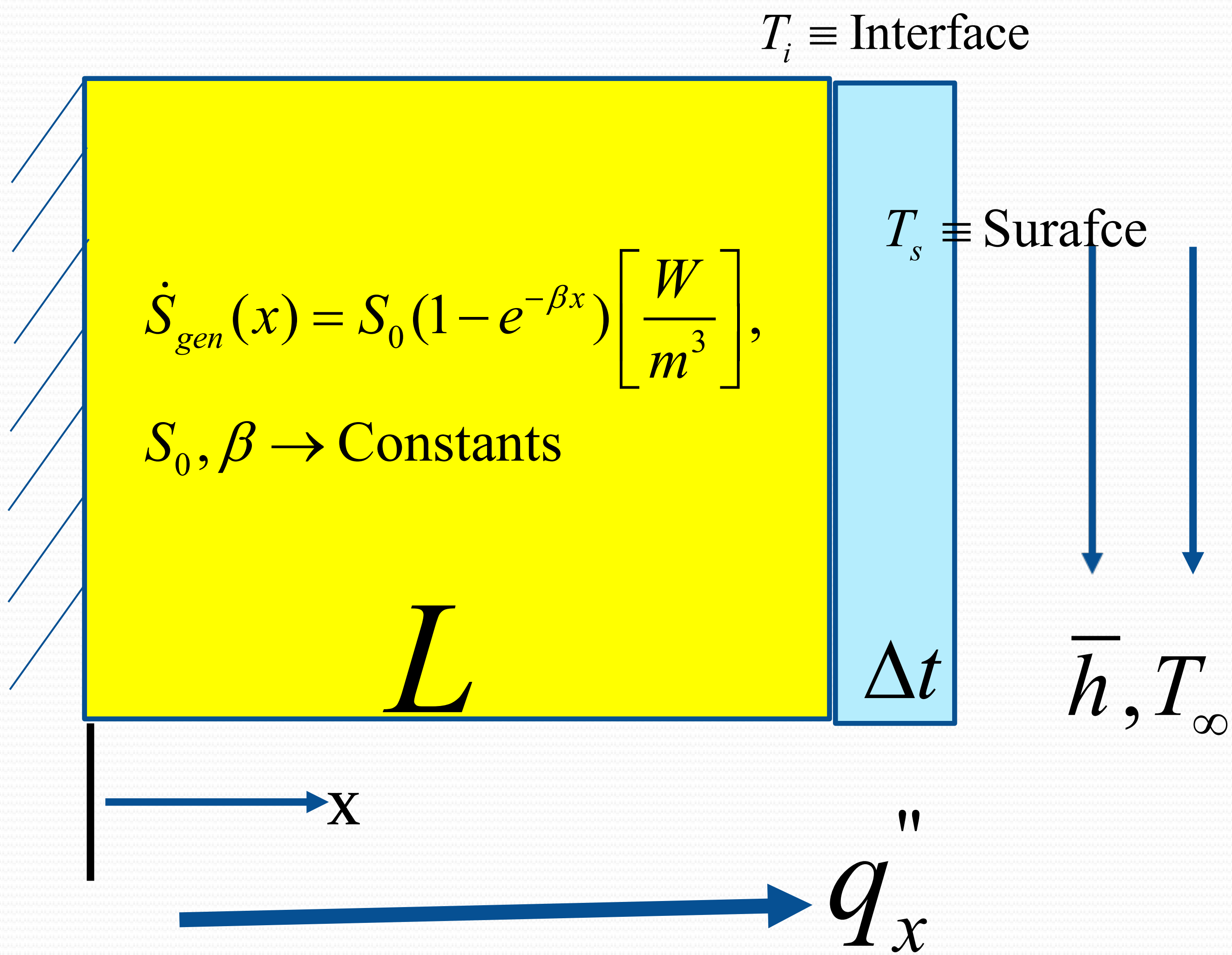


$T_i \equiv$ Interface
$T_s \equiv$ Surafce
$\dot{S}_{gen}(x) = S_0(1 - e^{-\beta x})\left[\frac{W}{m^3}\right],$
$S_0, \beta \rightarrow$ Constants
L
Δt
$\bar{h}, T_\infty$
x
q_x''

Steady State Temperature Formulation

BIO UNIT

HDE

$$\frac{d^2T_B}{dx^2} = -\frac{\dot{S}_{gen}(x)}{k_B} = -\frac{S_0(1-e^{-\beta x})\left[\frac{W}{m^3}\right]}{k_B}$$

$0 \le x \ge L$

Boundary Conditions

$$1.\left.\frac{dT_B}{dx}\right|_{x=0} = 0(insulation)$$

$$2.T_B(x=L) = T_I(x=0)$$

INSULATION

HDE

$$\frac{d^2T_I}{dx^2} = 0, 0 \le x \le \Delta t$$

Boundary Conditions

$$1.-k_I\left.\frac{dT_I}{dx}\right|_{x=0} = -k_B\left.\frac{dT_B}{dx}\right|_{x=L} \rightarrow$$

HEAT FLUX MUST BE SAME

$$2.-k_I\left.\frac{dT_I}{dx}\right|_{x=\Delta t} = h(T_I(x=\Delta t)-T_\infty)$$

NOTE: Sgen(x) is only being generated within BIO UNIT.
NOTE: Two (2) different COORDINATE SYSTEMS

BIO UNIT SOLUTION

MOST GENERAL SOLUTION

$0 \le x \le L$

$$\frac{d^2T_B}{dx^2} = -\frac{\dot{S}_{gen}(x)}{k_B} = -\frac{S_0(1-e^{-\beta x})\left[\frac{W}{m^3}\right]}{k_B}$$

Integrate

$$\frac{dT_B}{dx} = \frac{-S_0}{k_B}\left(x - \frac{e^{-\beta x}}{-\beta}\right) + C_1$$

BOUNDARY CONDITION - #1

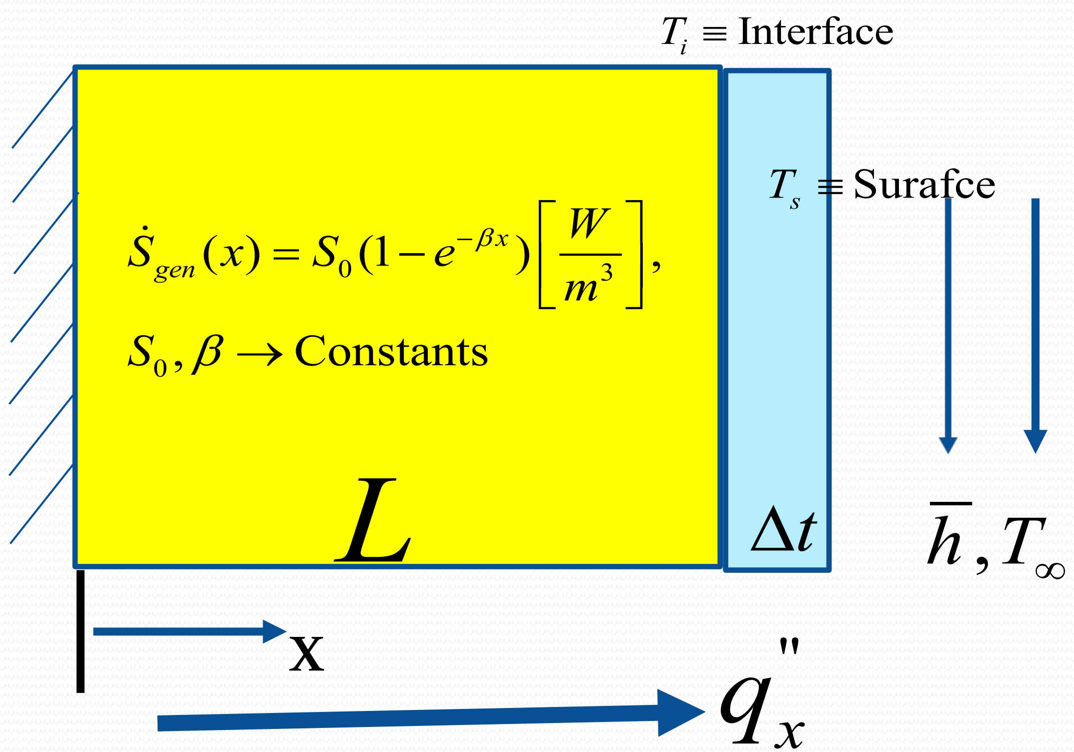
$$\left.\frac{dT_B}{dx}\right|_{x=0} = 0 = \frac{-S_0}{k_B}(0 - \frac{1}{-\beta}) + C_1$$

$$C_1 = \frac{S_0}{k_B\beta} \rightarrow \frac{\frac{W}{m^3}}{\frac{W}{m-K}\frac{1}{m}} \rightarrow \frac{K}{m}$$

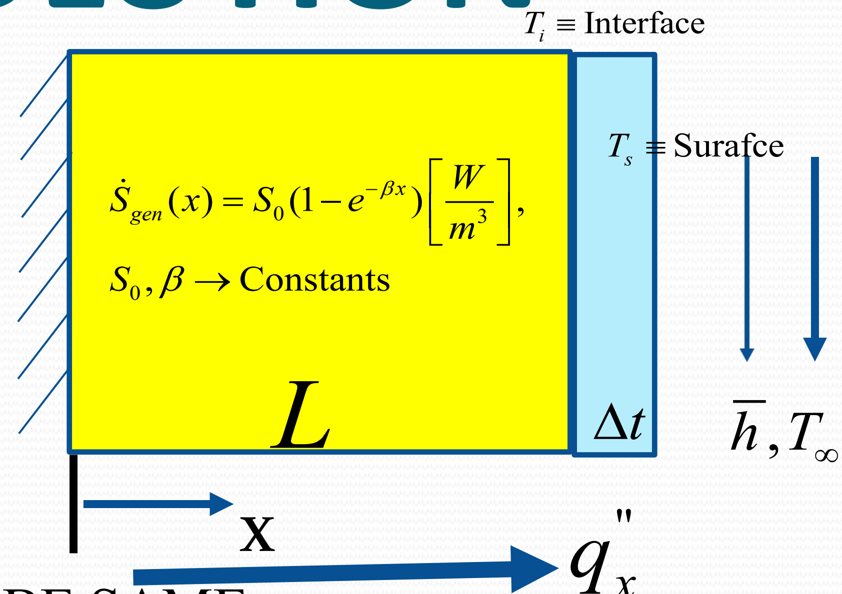
Integrate

$$T_B(x) = \frac{-S_0}{k_B}\left(\frac{x^2}{2} - \frac{e^{-\beta x}}{\beta^2}\right) + C_1 x + C_2$$

$$T_B(x) = \frac{-S_0}{k_B}\left(\frac{x^2}{2} - \frac{e^{-\beta x}}{\beta^2}\right) + \frac{S_0}{k_B\beta}x + C_2$$



INSULATION SOLUTION



$$\frac{d^2T_I}{dx^2} = 0, 0 \le x \le \Delta t$$

$$T_I(x) = D_1x + D_2$$

BOUNDARY CONDITIONS - #2/3/4

2. $T_I(x=0) = T_B(x=L) \rightarrow$ TEMPS MUST BE SAME

3. $-k_I \left.\frac{dT_I}{dx}\right|_{x=0} = -k_B \left.\frac{dT_B}{dx}\right|_{x=L} \rightarrow$ HEAT FLUX MUST BE SAME

4. $-k_I \left.\frac{dT_I}{dx}\right|_{x=\Delta t} = h(T_I(x=\Delta t) - T_\infty) \rightarrow$ CONDUCTION = CONVECTION

MOST GENERAL SOLUTION

$$T_I(x) = D_1x + D_2$$

BOUNDARY CONDITIONS

3 UNKNOWNS

C_2, D_1, D_2

3 BOUNDARY CONDITIONS

EQUATE TWO SOLUTIONS AT INTERFACE

BIO UNIT $\rightarrow$

$$T_B(x) = \frac{-S_0}{k_B}\left(\frac{x^2}{2} - \frac{e^{-\beta x}}{\beta^2}\right) + \frac{S_0}{k_B \beta} x + C_2 \rightarrow 0 \le x \le L$$

LEAD INSULATION $\rightarrow$

$$T_I(x) = D_1 x + D_2 \rightarrow 0 \le x \le \Delta t$$

EQUATE - BC#2

$T_I(x=0) = T_B(x=L) \rightarrow$ TEMPS MUST BE SAME

$$D_2 = \frac{-S_0}{k_B}\left(\frac{L^2}{2} - \frac{e^{-\beta L}}{\beta^2}\right) + \frac{S_0}{k_B \beta} L + C_2$$

BC#3

$$-k_I \left.\frac{dT_I}{dx}\right|_{x=0} = -k_B \left.\frac{dT_B}{dx}\right|_{x=L} \rightarrow \text{HEAT FLUX MUST BE SAME}$$

$$(-k_I) D_1 = -k_b \left(\frac{-S_0}{k_B}\left(L - \frac{e^{-\beta L}}{-\beta}\right) + \frac{S_0}{k_B \beta}\right)$$

$$D_1 = \frac{-\left(-S_0\left(L - \frac{e^{-\beta L}}{-\beta}\right) + \frac{S_0}{\beta}\right)}{(-k_I)}$$

$$C_2 = D_2 + \frac{S_0}{k_B}\left(\frac{L^2}{2} - \frac{e^{-\beta L}}{\beta^2}\right) - \frac{S_0}{k_B \beta} L$$

Boundary Condition #4

At Outer Boundary:

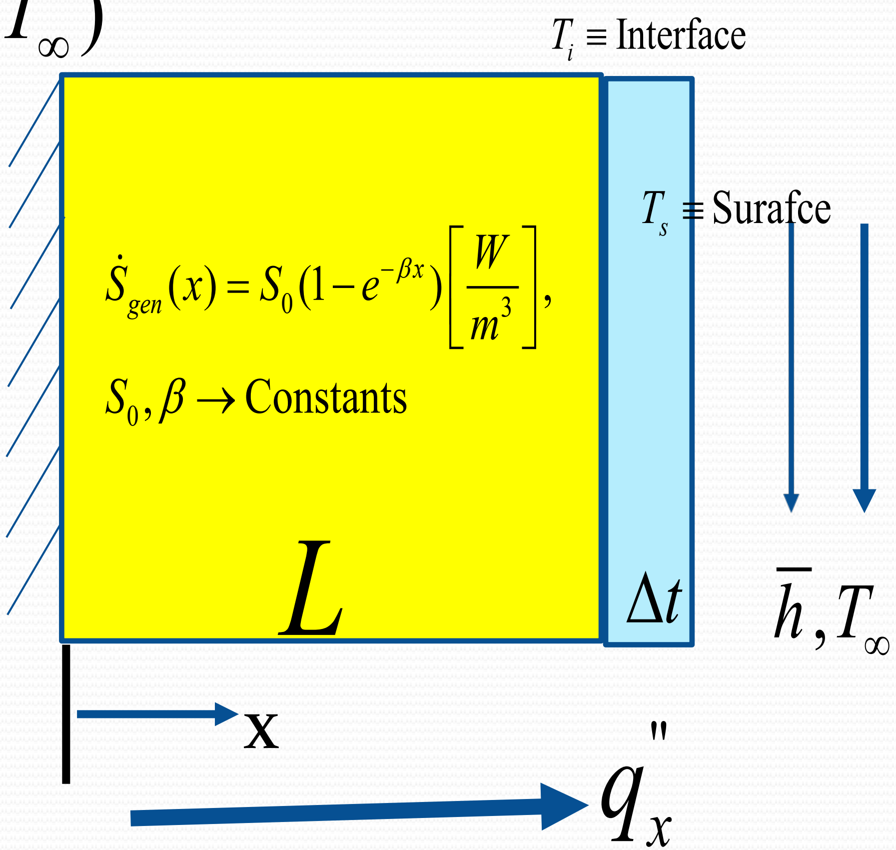
$$q_x''(conduction) = q_s''(convection)$$

$$-k_I \left.\frac{dT_I}{dx}\right|_{x=\Delta t} = h(T(x=\Delta t) - T_\infty)$$

$$-k_I D_1 = h(D_1 \Delta t + D_2 - T_\infty)$$

$$D_2 = \frac{-k_I D_1}{h} - D_1 \Delta t + T_\infty$$

$$D_2 = -D_1\left(\frac{k_I}{h} + \Delta t\right) + T_\infty$$



SUMMARY CONTANTS

BIO UNIT $\rightarrow$

$$T_B(x) = \frac{-S_0}{k_B}\left(\frac{x^2}{2} - \frac{e^{-\beta x}}{\beta^2}\right) + \frac{S_0}{k_B \beta} x + C_2 \rightarrow 0 \le x \le L$$

LEAD INSULATION $\rightarrow$

$$T_I(x) = D_1 x + D_2 \rightarrow 0 \le x \le \Delta t$$

$$C_1 = \frac{S_0}{k_B \beta}$$

$$D_1 = \frac{-(-S_0\left(L - \frac{e^{-\beta L}}{-\beta}\right) + \frac{S_0}{\beta})}{(-k_I)} = \Omega$$

$$D_2 = -\Omega\left(\frac{k_I}{h} + \Delta t\right) + T_\infty$$

$$C_2 = D_2 + \frac{S_0}{k_B}\left(\frac{L^2}{2} - \frac{e^{-\beta L}}{\beta^2}\right) - \frac{S_0}{k_B \beta} L$$

MATRIX FORMAT (Maple/Matlab)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{S_0}{k_B \beta} \\ \dfrac{S_0}{k_B}\left(\dfrac{L^2}{2} - \dfrac{e^{-\beta L}}{\beta^2}\right) - \dfrac{S_0}{k_B \beta} L \\ \Omega \\ -\Omega\left(\dfrac{k_I}{h} + \Delta t\right) + T_\infty \end{Bmatrix}$$

UNIT CHECK (INSULATION)

$$T_I(x) = D_1 x + D_2 \rightarrow 0 \leq x \leq \Delta t$$

$$D_1 = \frac{-(-S_0\left[\frac{W}{m^3}\right]\left(L[m] - \frac{e^{-\beta L}}{-\beta\left[\frac{1}{m}\right]}\right) + \frac{S_0\left[\frac{W}{m^3}\right]}{\beta\left[\frac{1}{m}\right]})}{(-k_I\left[\frac{W}{m-K}\right])} = \left[\frac{K}{m}\right]$$

$$D_2 = -D_1\left[\frac{K}{m}\right]\left(\frac{k_I\left[\frac{W}{m-K}\right]}{h\left[\frac{W}{m^2-K}\right]} + \Delta t[m]\right) + T_\infty[K] = [K]$$

UNIT CHECK (BIO UNIT)

$$T_B(x) = \frac{-S_0}{k_B}\left(\frac{x^2}{2} - \frac{e^{-\beta x}}{\beta^2}\right) + \frac{S_0}{k_B \beta} x + C_2 \rightarrow 0 \le x \le L$$

$$C_1 = \frac{S_0\left[\frac{W}{m^3}\right]}{k_B\left[\frac{W}{m-K}\right]\beta\left[\frac{1}{m}\right]} = \left[\frac{K}{m}\right]$$

$$C_2 = D_2[K] + \frac{S_0\left[\frac{W}{m^3}\right]}{k_B\left[\frac{W}{m-K}\right]}\left(\frac{L^2\left[m^2\right]}{2} - \frac{e^{-\beta L}}{\beta^2\left[\frac{1}{m^2}\right]}\right) - \frac{S_0\left[\frac{W}{m^3}\right]}{k_B\left[\frac{W}{m-K}\right]\beta\left[\frac{1}{m}\right]} L = [K]$$

INTERFACE AND SURFACE TEMPERATURES

INTERFACE TEMPERATURE

BIO UNIT $\rightarrow$

$$T_B(x) = \frac{-S_0}{k_B}\left(\frac{x^2}{2} - \frac{e^{-\beta x}}{\beta^2}\right) + \frac{S_0}{k_B \beta} x + C_2 \rightarrow 0 \le x \le L$$

let x=L

$$\mathrm{T}_B(x = L) = \frac{-S_0}{k_B}\left(\frac{L^2}{2} - \frac{e^{-\beta L}}{\beta^2}\right) + \frac{S_0}{k_B \beta} L + C_2$$

SURFACE TEMPERATURE

LEAD INSULATION $\rightarrow$

$$T_I(x) = D_1 x + D_2 \rightarrow 0 \le x \le \Delta t$$

let x=Δt

$$T_I(x = \Delta t) = D_1 \Delta t + D_2$$

BIO UNIT "WALL" TEMPERATURE

$$T_B(x) = \frac{-S_0}{k_B}\left(\frac{x^2}{2} - \frac{e^{-\beta x}}{\beta^2}\right) + \frac{S_0}{k_B\beta}x + C_2 \rightarrow 0 \le x \le L$$

Let x=0

$$\mathrm{T}_B(x=0) = T_{wall} = \frac{-S_0}{k_B}(-\frac{1}{\beta}) + C_2$$

$$D_2 = -\Omega(\frac{k_I}{h} + \Delta t) + T_\infty$$

$$C_2 = D_2 + \frac{S_0}{k_B}\left(\frac{L^2}{2} - \frac{e^{-\beta L}}{\beta^2}\right) - \frac{S_0}{k_B\beta}L$$

2nd Method

INTERFACE and SURFACE TEMPERATURE

SURFACE TEMPERATURE, Ts = ?

$$\dot{S}_{gen}(x) = S_0(1-e^{-\beta x})\left[\frac{W}{m^3}\right], S_0, \beta \rightarrow \text{Constants}, 0 \le x \le L$$

Overall Energy Balance at SS (BIO UNIT + INSULATION)

$$\dot{E}_{gen} = \dot{E}_{out} = hA_s(T_s - T_\infty)$$

$$T_s = \frac{\dot{E}_{gen}}{hA_s} + T_\infty = \frac{\dot{E}_{gen}}{h(HD)} + T_\infty$$

$$\dot{E}_{gen} = \int \dot{S}_{gen}(x)dV, dV = HDdx$$

$$\dot{E}_{gen} = HDS_0\int_0^L (1-e^{-\beta x})dx = HDS_0(x - \frac{e^{-\beta x}}{-\beta})_{0-L}$$

$$\dot{E}_{gen} = HDS_0(L - \frac{e^{-\beta L}}{-\beta} - \frac{1}{\beta}) = HDS_0(L - \frac{1}{\beta}(e^{-\beta L} + 1))$$

$$T_s = \frac{\dot{E}_{gen}}{h(HD)} + T_\infty = \frac{(HD)S_0(L - \frac{1}{\beta}(e^{-\beta L}+1))}{h(HD)} + T_\infty$$

INTERFACE TEMPERATURE (Lead Insulation Energy Balance)

$$-k_I A_c \frac{dT}{dx}$$

$$hA_s(T_s - T_\infty)$$

EGEN (W)

q(W) Conduction

q (W) Convection

Ti

$1D$

Steady State

$\dot{S}_{gen} = 0$

Ts

$$Egen = -k_I Ac \frac{dT}{dx} = \frac{T_1 - T_2}{\frac{L}{k_I Ac}}$$

$$\frac{dT}{dx} = \frac{(T_s - T_i)}{\Delta t} = \frac{Egen}{-k_I Ac}$$

$$T_i = T_s + \frac{Egen[W] \bullet \Delta t[m]}{k_I \left[\frac{W}{m-K}\right] Ac\left[m^2\right]}$$

$$T_s = \frac{\dot{E}_{gen}}{hA_s} + T_\infty = \frac{\dot{E}_{gen}}{h(HD)} + T_\infty$$

NOTE!!!!!!!!!

2nd Method ONLY provides interface and surface temperatures.

Does NOT provide internal temperature distributions within the media.

HIGH LEVEL ROADMAP

Solve HDE for each HT Media with two Boundary Conditions each, 4 total.

Each HDE solution must have individual constant names (4 constants) to determine EXACT solution for temperature EVERWHERE within media.

Match Temperature and Heat Flux at interface between HT media using two (2) boundary conditions.

I can always find heat flux and heat rate by via FOURIER's LAW!!!

Use boundary conditions at interface and outer conditions to solve for 4 constants.

Check units on constants to ensure no slip-ups on algebra.

Seek to apply OVERALL control volume (if heat FLUX is known at every boundary) to determine an outer temperature. This can be used as "one" of your boundary conditions if desired.