Hydrostatic Force on a Submerged Plane Surface

ZOOM LECTURE LINK

https://kettering-edu.zoom.us/my/drberry.heat?pwd=STlNVWYwMktvRUdXSWJsSGFSK2RPZz09

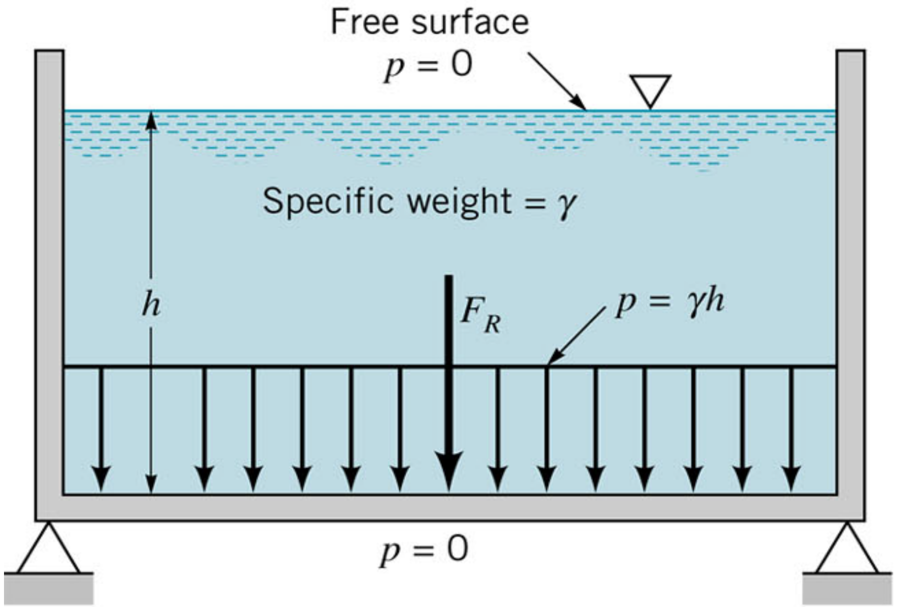
OFFICE HOURS
R 8:30PM – 9:30PM

Hydrostatic Force: Case I - Horizontal Surface

The magnitude of the resultant force is simply:

$$F_R = pA$$

$p = uniform\ pressure\ at\ the\ bottom$
$A = area\ of\ the\ bottom.$



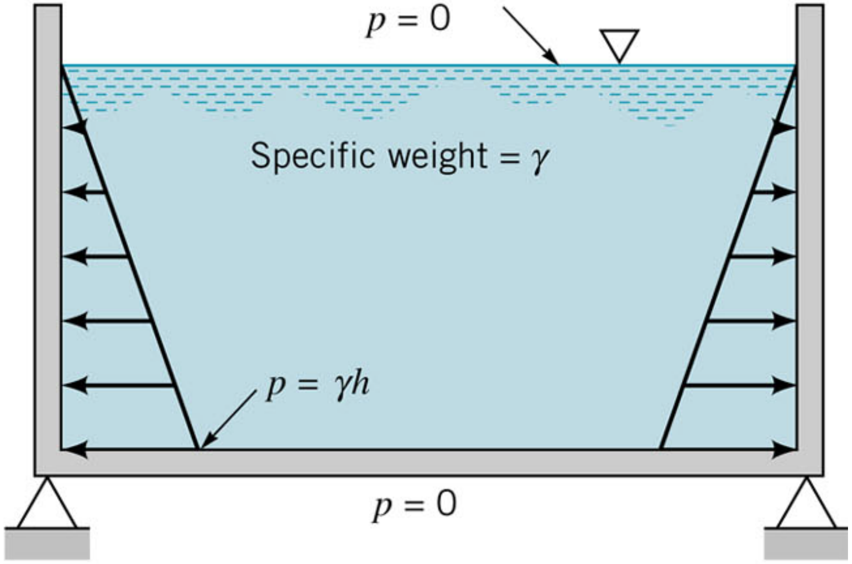
(*a*) Pressure on tank bottom

Observation: **Since the pressure is constant and uniformly distributed over the bottom, the resultant force acts through the Centroid of the area.**

Hydrostatic Force: Case II - Ends of an open tank

The pressure on the ends of the tank is not uniformly distributed.

Determination of the resultant force for this situation is different.



(b) Pressure on tank ends

Observation: **The resultant force of a static fluid on a plane surface is due to the Hydrostatic pressure distribution on the surface.**

Hydrostatic Force on a Plane Surface

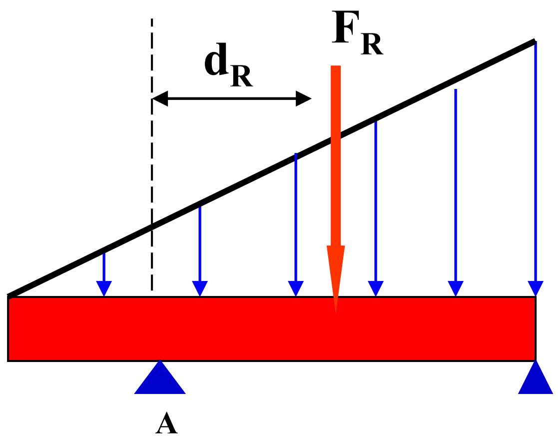
Class 05: Equilibrium Force System

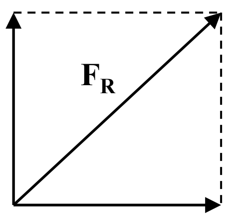
*** Statics***

- Distributed Force Systems
- First Moment of Inertia
- Second Moment of Inertia
- Parallel axis theorem
- Moment about a point
- Moment about an axis
- Centroid – Center of Gravity

Class 05: Distributed Forces and Static Equilibrium

Distributed forces acting on a surface





How do we handle this situation? → **Equivalent Force System**

$$\leftrightarrow \sum F_x = 0;$$

$$\updownarrow \sum F_y = 0;$$

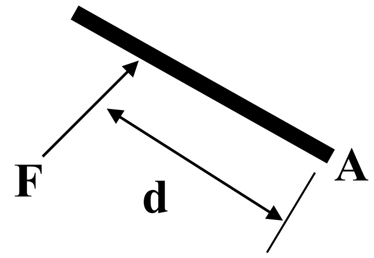
$$_{CC/CW} \sum M_A = 0;$$

Force: $F_R = \int dF$

Moment: $F_R \cdot d_R = \int x \, dF$

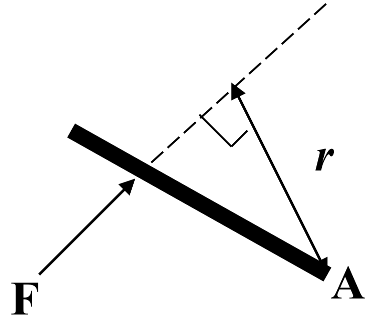
Distributed Forces & Static Equilibrium

Moment:



d: perpendicular distance from the line of action to the point A.

Moment about point A: $M_A = F \cdot d$



$\vec{r}$**: A to F vector.**

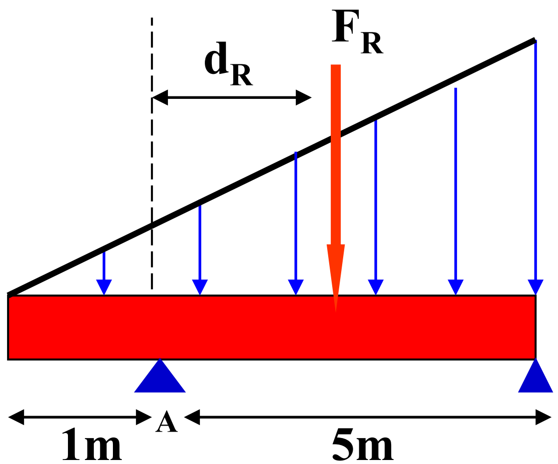
Moment about point A: $\vec{M}_A = \vec{r} \times \vec{F}$

Class 05: Distributed Forces and Static Equilibrium

Example: **If**

$$F(x) = 2\frac{N}{m^2} \bullet x;$$

$$Find\ d_R = ?$$



$$F(x) = 2\frac{N}{m^2} \bullet x$$

$$F_R = \int_0^L 2\frac{N}{m^2} \bullet x\,dx = \frac{2x^2}{2}\bigg|_{0-6} = 36N$$

Solution: **Take moment about point A:**

$$F_R \cdot d_R = \int x\,dF$$

$$\Rightarrow (36N)d_R = \int_{-1m}^{5m} x\left(2\frac{N}{m^2}x\right)dx$$

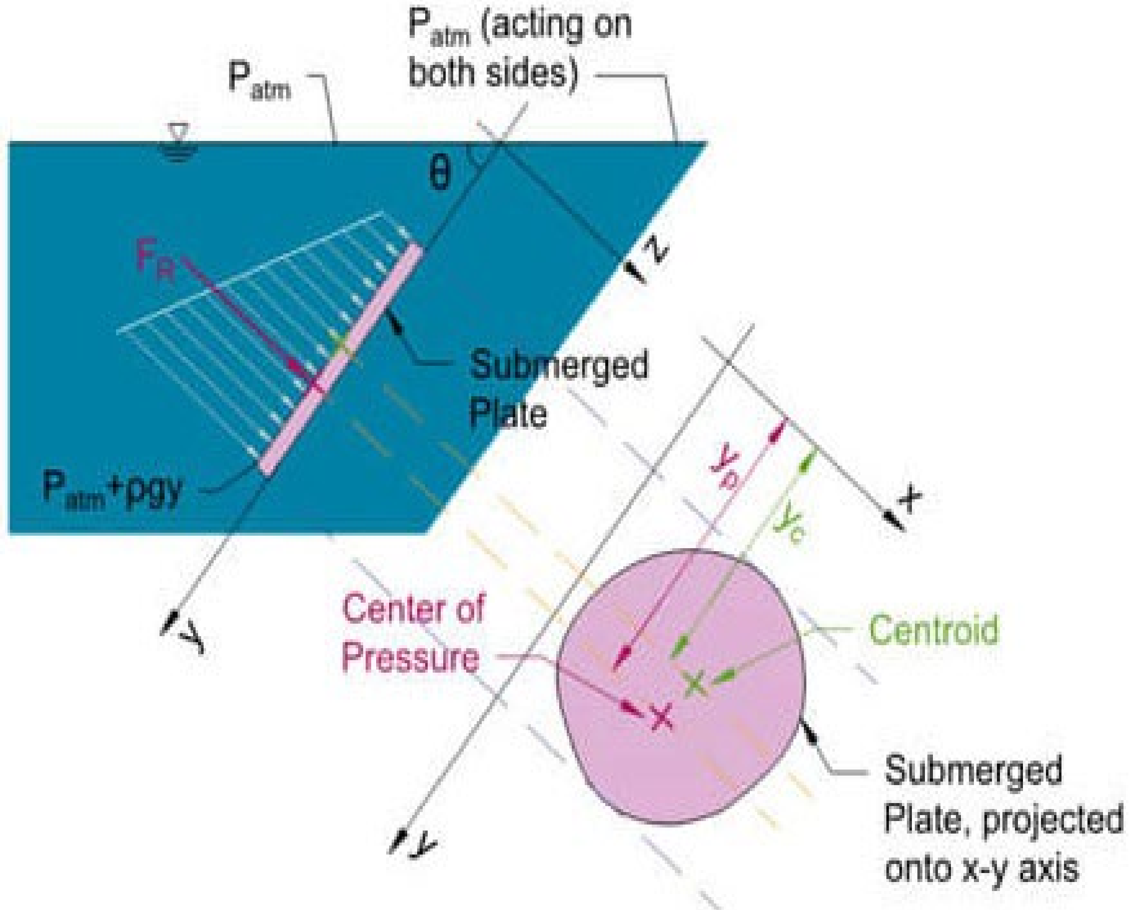
$$= \left(2\frac{N}{m^2}\right)\int_{-1m}^{5m} x^2\,dx$$

$$= \left(2\frac{N}{m^2}\right)\left(\frac{x^3}{3}\right)\bigg|_{-1m}^{5m} = 82.6(N.m)$$

$$\Rightarrow d_R = 2.3m$$

- **Location of the resultant force is right to point A – just less than the ½ way from A to the edge.**

Hydrostatic Force on a Plane SUBMERGED Surface



PASCAL's LAW

Observation: The pressure at a point in a fluid at rest or in motion, is independent of direction as long as there are no shearing stresses present.

LAW OF HYDROSTATICS

$$\frac{dP}{dz} = -\gamma_{fluid} \quad \uparrow Z \quad \downarrow g$$

LAW OF HYDROSTATICS

$$\frac{dP}{dz} = +\gamma_{fluid} \quad \downarrow Z \quad \downarrow g$$

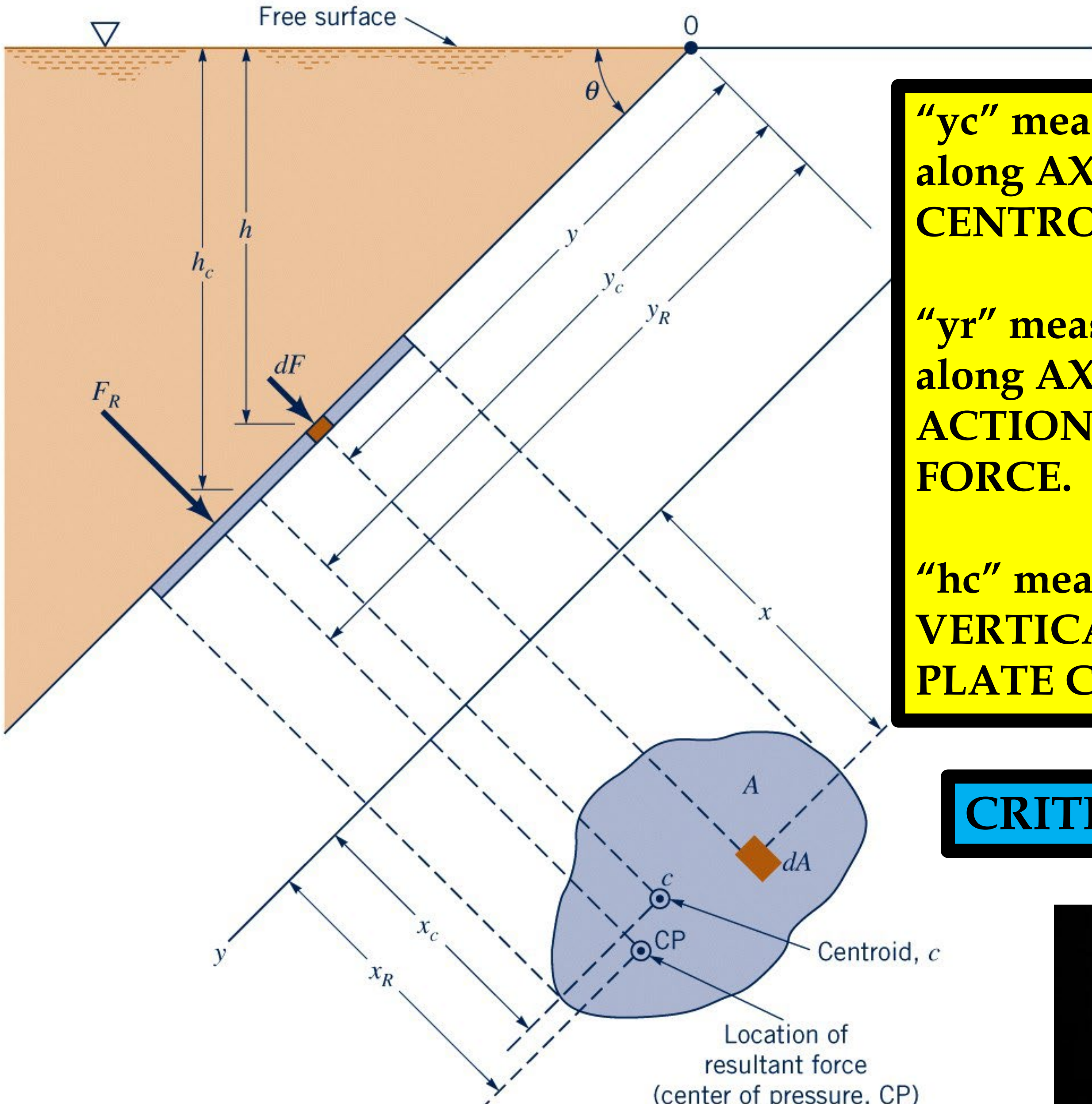


Figure 2.17

"yc" measured from SURFACE along AXIS of plate-to-plate CENTROID.

"yr" measured from SURFACE along AXIS of plate to LINE of ACTION of resultant pressure FORCE.

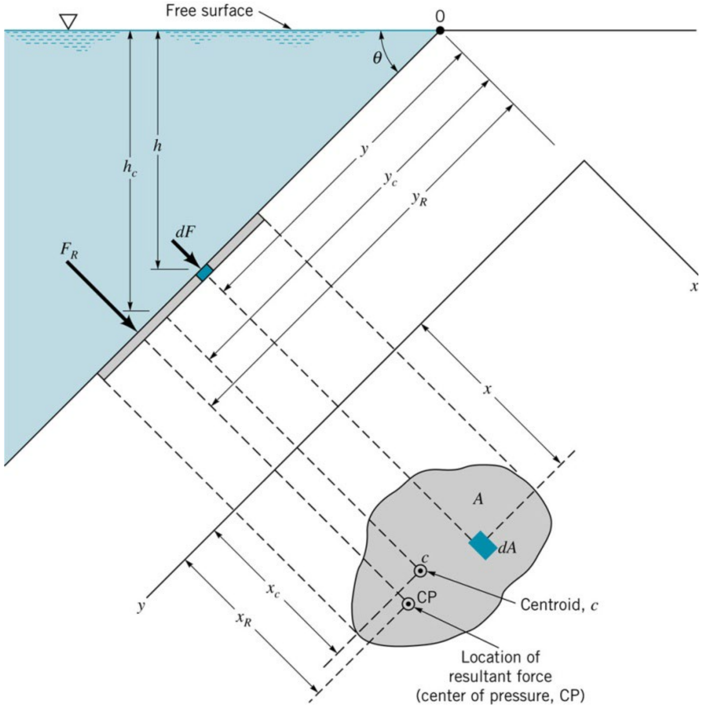
"hc" measured from SURFACE VERTICALLY to location of PLATE CENTROID.

CRITICAL DEFINITIONS

Class 05: Hydrostatic Force on a Submerged Surface

• Determination of resultant force acting on submerged bodies

✓First consider a planar arbitrary shape submerged in a liquid (see Fig.). The plane makes an angle θ with the liquid surface. The depth of water over the plane varies linearly.



✓Now prescribe a coordinate frame such that the y-axis is aligned with the submerged plane. Consider an infinitesimally small area, ***dA=dx.dy*** at a ***(x,y)***. Let this small area be located at a depth ***h*** from the free surface.

✓We know

$$p = p_a + \gamma h \qquad (1)$$

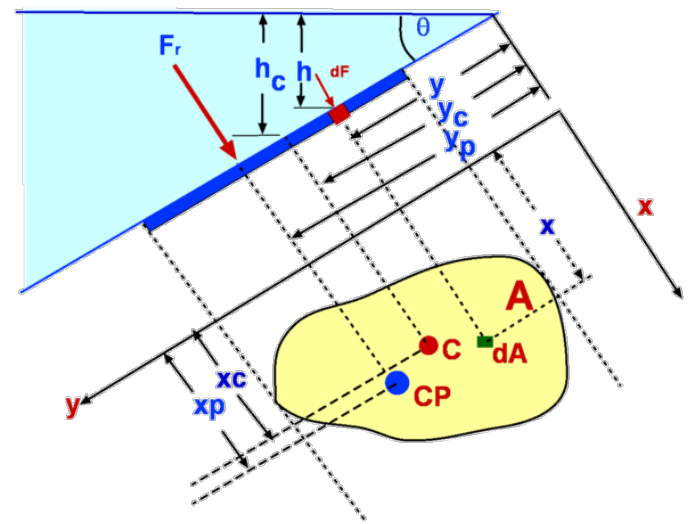
Class 05: Hydrostatic Force on a Submerged Surface

The hydrostatic force on the plane is given by

$$F_R = \int p\, dA = \int (p_a + \gamma h)\, dA$$

$$= p_a A + \gamma \int_A h\, dA = p_a A + \gamma \int_A y \sin\theta\, dA$$

$$= p_a A + \gamma \sin\theta \int_A y\, dA \qquad (2)$$



First moment?

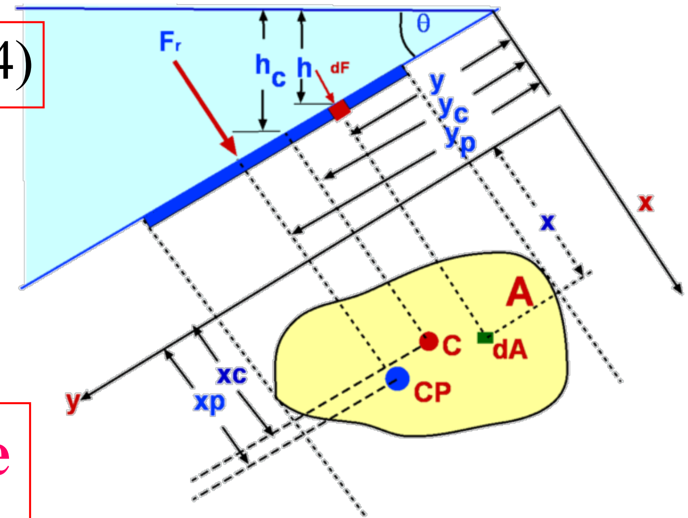
The integral, $\int_A y\, dA$ is the first moment of surface area about x-axis. If, y_c is the centroid of the area, we have

$$\int_A y\, dA = y_c \text{ A} \qquad (3)$$

Class 05: Hydrostatic Force on a Submerged Surface

From (2) and (3), we obtain

$$F_R = p_a A + (\gamma \sin\theta)(y_c A) = p_a A + \gamma h_c A = P_c A \quad (4)$$



$h_c = y_c \sin\theta$ and

$P_c = P_a + \gamma h_c$ is pressure acting at centroid

Observation: The magnitude of the resultant fluid force is equal to the pressure acting at the centroid of the area multiplied by the total area

Note: Even though the force can be computed from the pressure at the center of the plane (centroid of the plane), this is **NOT the point through which the Force acts!**

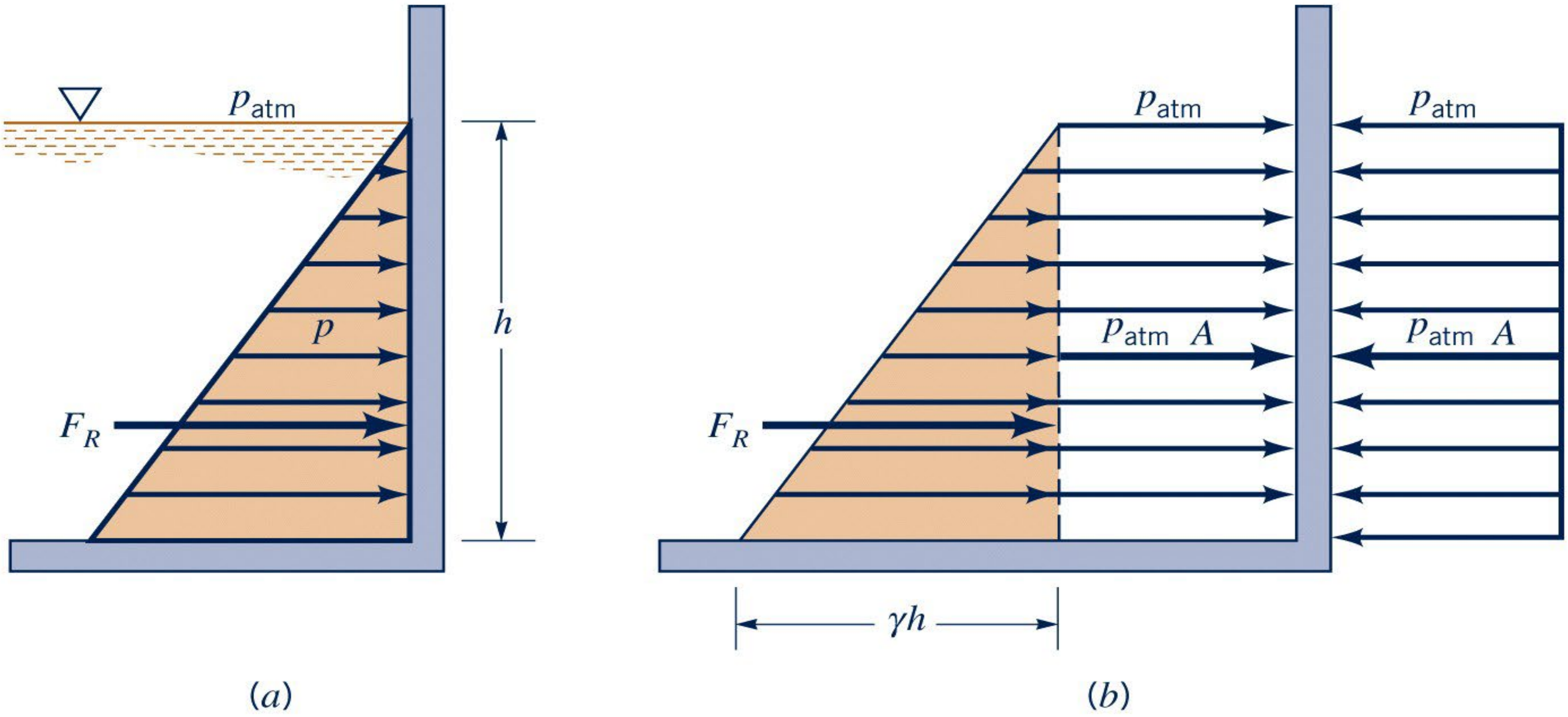


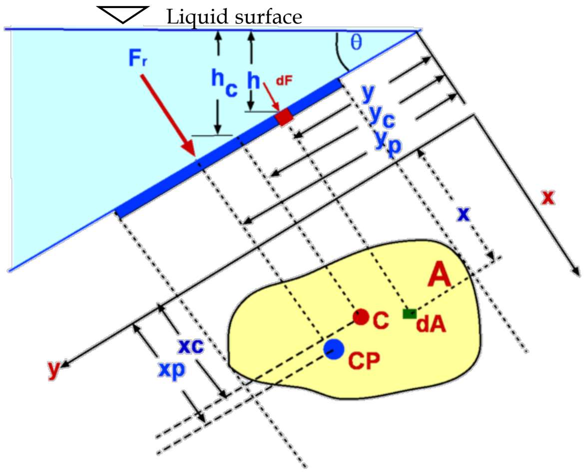
Figure 2.22

$$F_R = p_a A + (\gamma \sin\theta)(\,y_c\, A) = p_a A + \gamma h_c\, A = P_c\, A \qquad (4)$$

We know that the atmospheric pressure p_a acting at the free surface also acts everywhere within the fluid and also on both sides of the plane. As such it doesn't contribute to the net force on the plane. So we can drop this term from Eq. (4) for F_R.

Class 05: Location of Hydrostatic Force

Location of the resultant force - (x_R, y_R)

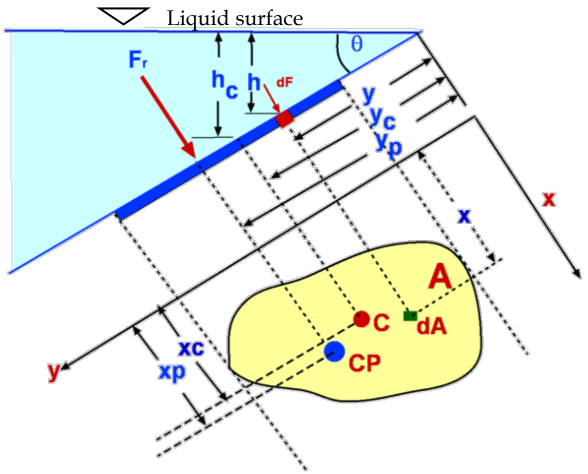


Force Balance: The moment of the resultant force must be equal to the moment of the pressure distributed force about the same axis, thus we have

$$x_R F_R = \int_A x\, p\, dA, \qquad y_R F_R = \int_A y\, p\, dA \qquad (5)$$

Class 05: Location of Hydrostatic Force

The term $\int_A y^2 dA$ is well-known second moment of the area (moment of inertia) about the x-axis and denoted by I_{xx}



$$y_R (F_R) = \int_A y \, (p) \, dA \quad (5)$$

$$y_R (\gamma \sin\theta \, y_c \, A) = \int_A y \, (\gamma h) \, dA = \int_A y \, \gamma (y \sin\theta) dA = \gamma \sin\theta \int_A y^2 dA$$

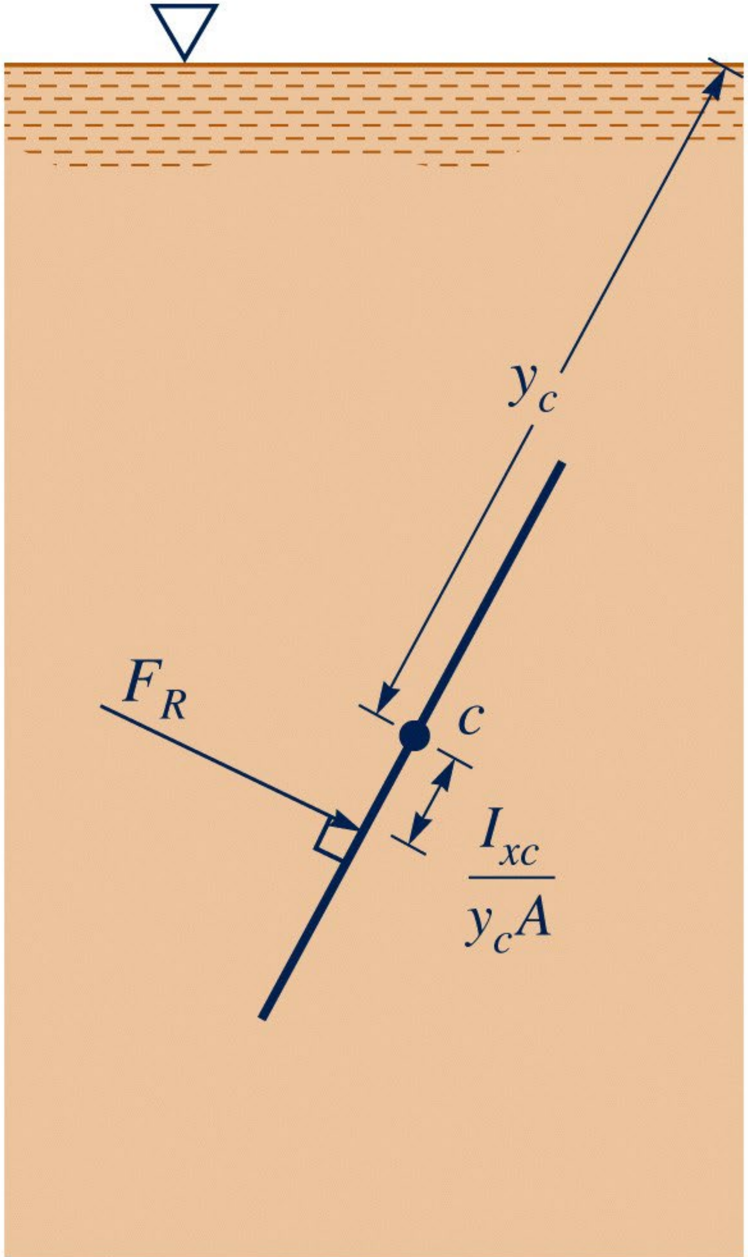
$$\Rightarrow y_R = \frac{\int_A y^2 dA}{y_c A} = \frac{Ixx}{y_c A} \quad (6)$$

Is the second moment of the area with respect to an axis passing through its centroid and parallel to the x-axis.

Using parallel axis theorem:

$$I_{xx} = I_{xc} + A y_c^{\ 2} \qquad (7)$$

$$y_R = \frac{I_{xc}}{y_c A} + y_c \qquad (8)$$



Unnumbered 2 p61b

Observation: Resultant force doesn't pass through the centroid but is always below it, since

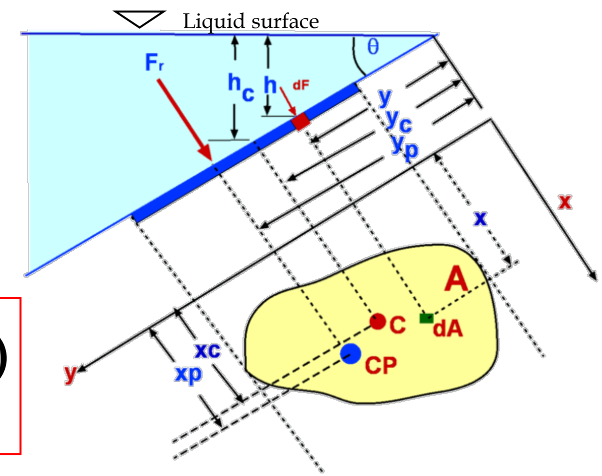
$$I_{xc}/y_c A > 0.$$

$$y_R = \frac{I_{xc}}{y_c A} + y_c \qquad (8)$$

Class 05: Location of Hydrostatic Force

The x-coordinate, x_R, for the resultant force can be determined in a similar manner by summing moments about the y-axis. Thus

$$x_R F_R = \int_A x\, p\, dA = \gamma \sin\theta \int_A xy dA \qquad (9)$$



Hence,

$$x_R = \frac{\int_A xy\, dA}{y_c A} = \frac{I_{xy}}{y_c A} \qquad (10)$$

The term I_{xy} is the product of inertia with respect to the x and y axes.

Using parallel axis theorem we can write

$$x_R = \frac{I_{xyc}}{y_c A} + x_c \qquad (11)$$

where I_{xyc} is the product of inertia with respect to an orthogonal coordinate system passing through the centroid of the area and formed by a translation of the x-y coordinate system.

Observation: The point through which the resultant force acts is called the *center of pressure*

Hydrostatic Force on a Submerged Plane Surface
"REVIEW"

Dealing with Hydrostatic Force

- **Magnitude of Equivalent Force (Fr)**

  Pressure at the depth of the centroid (y_c)

- **Locate the Equivalent Force**

  Equate resultant moment to distributed moment (y_R)

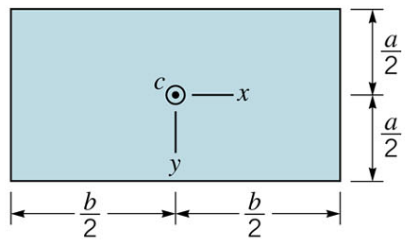
- **Apply Equilibrium Conditions**

  Complete the rest of the problem to find REACTION forces.

Pay Attention!

Although you found F_R by determining the pressure at the centroid, DO NOT LOCATE F_R at that point. F_R LOCATED AT y_R (i.e. at y_{CP}).

Class 05: Geometrical properties of some common shapes



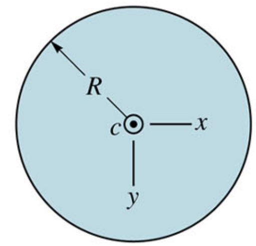
$A = ba$

$I_{xc} = \frac{1}{12} ba^3$

$I_{yc} = \frac{1}{12} ab^3$

$I_{xyc} = 0$

(*a*) Rectangle

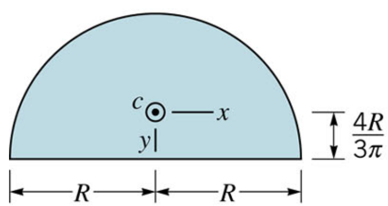


$A = \pi R^2$

$I_{xc} = I_{yc} = \frac{\pi R^4}{4}$

$I_{xyc} = 0$

(*b*) Circle



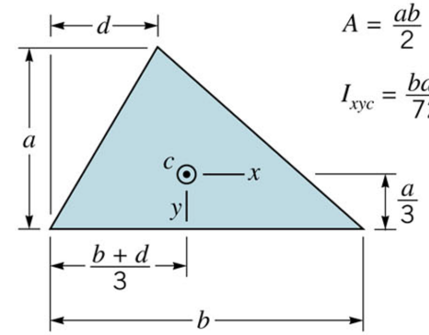
$A = \frac{\pi R^2}{2}$

$I_{xc} = 0.1098 R^4$

$I_{yc} = 0.3927 R^4$

$I_{xyc} = 0$

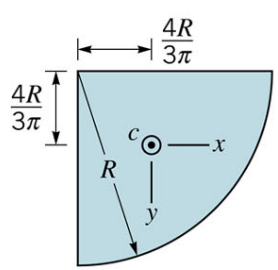
(*c*) Semicircle



$A = \frac{ab}{2}$ $\quad I_{xc} = \frac{ba^3}{36}$

$I_{xyc} = \frac{ba^2}{72}(b - 2d)$

(*d*) Triangle



$A = \frac{\pi R^2}{4}$

$I_{xc} = I_{yc} = 0.05488 R^4$

$I_{xyc} = -0.01647 R^4$

(*e*) Quarter circle

NOTHING BUT THE MATH

ENGINEERING

$$F_R = \gamma_f h_c A$$

$$y_r = y_c + \frac{I_{xxc}}{y_c A_p}$$

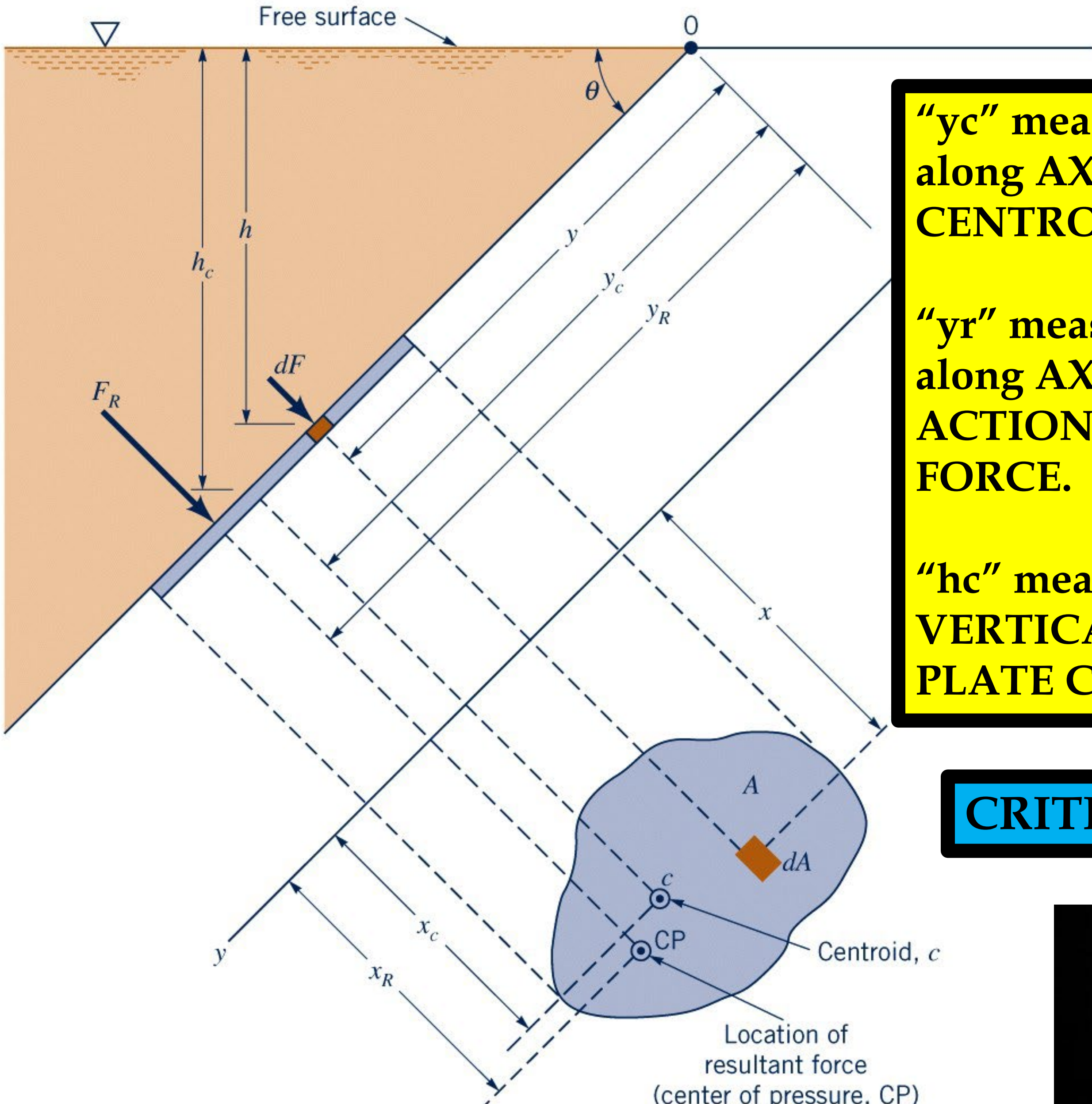


Figure 2.17

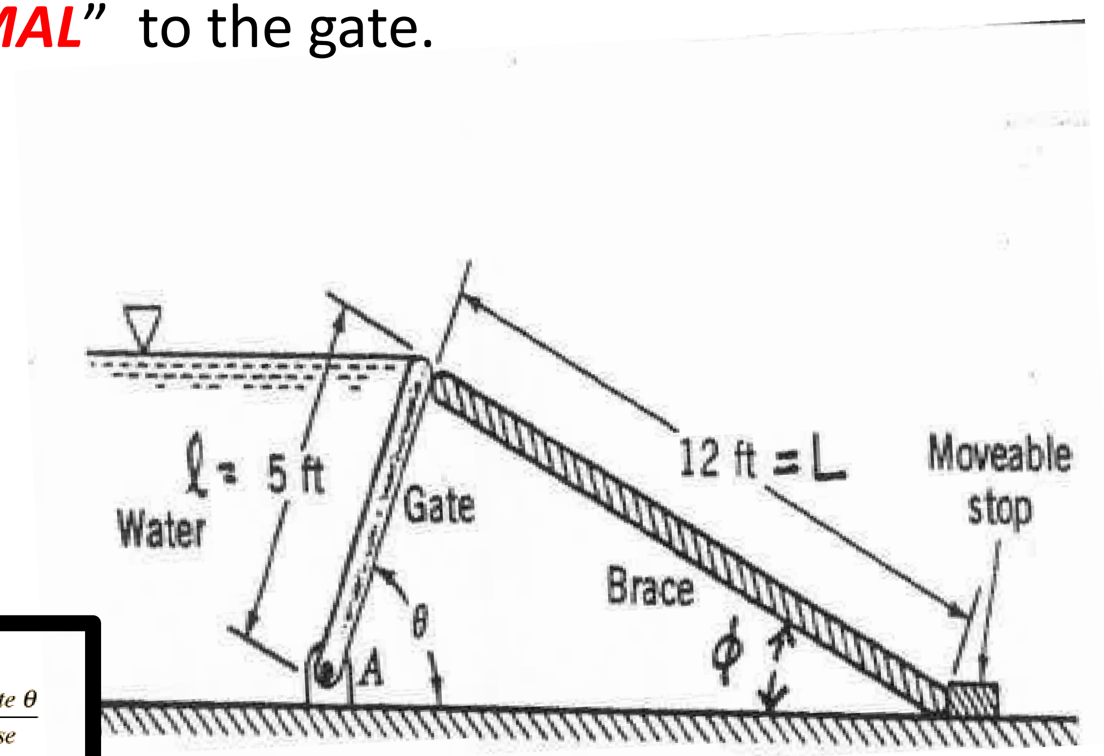
"yc" measured from SURFACE along AXIS of plate-to-plate CENTROID.

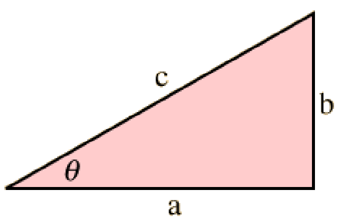
"yr" measured from SURFACE along AXIS of plate to LINE of ACTION of resultant pressure FORCE.

"hc" measured from SURFACE VERTICALLY to location of PLATE CENTROID.

CRITICAL DEFINITIONS

A 200 lb gate of 10ft wide and 5 ft. long is hinged at point A as shown. The gate is held in place by a brace that acts "***NORMAL***" to the gate.





$$\text{sine}: \sin\theta = \frac{b}{c} = \frac{\textit{side opposite } \theta}{\textit{hypotenuse}}$$

$$\text{cosine}: \cos\theta = \frac{a}{c} = \frac{\textit{side adjacent } \theta}{\textit{hypotenuse}}$$

$$\text{tangent}: \tan\theta = \frac{b}{a} = \frac{\textit{side opposite } \theta}{\textit{side adjacent } \theta} = \frac{\sin\theta}{\cos\theta}$$

$$\text{cotangent}: \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

$$\text{secant}: \sec\theta = \frac{1}{\cos\theta}$$

$$\text{cosecant}: \csc\theta = \frac{1}{\sin\theta}$$

DRAW a complete free Body Diagram of gate and show all forces, assumed COORDINATE SYSTEM, and indicate "yc", "yr", and "hc".

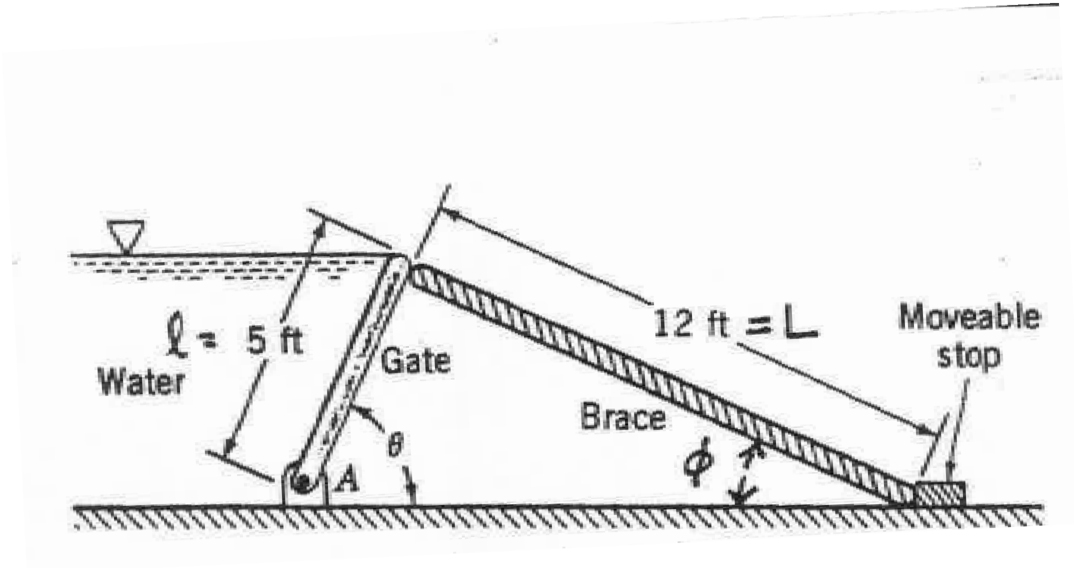
Determine:

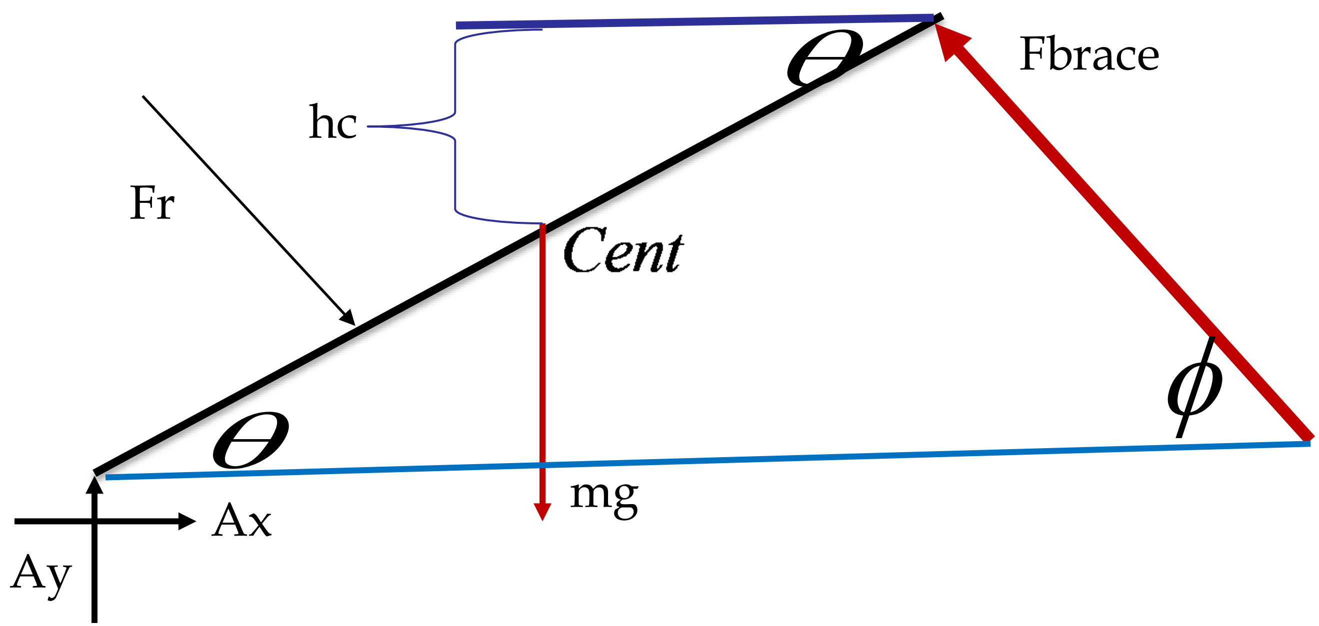
a. The resultant pressure force AND location?

b. The force of the brace on the gate?

c. Derive the parametric equation/expression to find the hinge forces at point A and verify units.

FREE BODY DIAGRAM





Resultant Fluid Pressure Force

"hc" measured from SURFACE VERTICALLY to location of PLATE CENTROID.

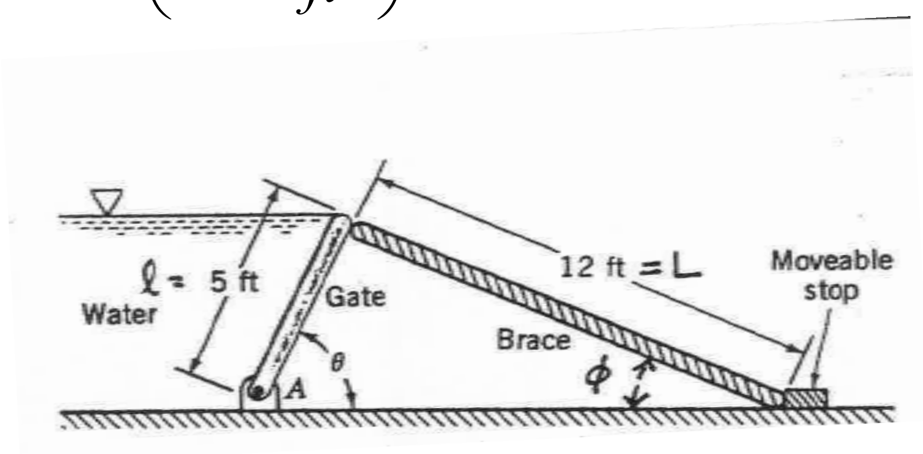
$$F_r = \gamma h_c A_p,$$

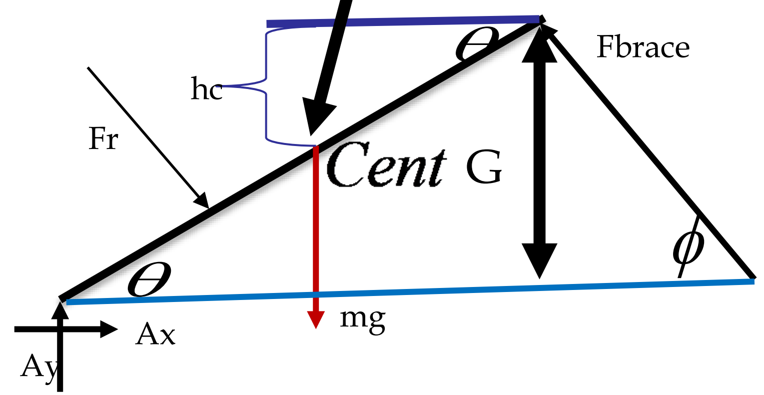
$$\theta = \tan^{-1}\frac{12}{5} = 67.4^0, \phi = 90 - \theta = 22.6^0$$

$$G \equiv \text{Verticle Height} = (l \sin\theta)$$

$$h_c = \frac{G}{2} = \frac{5\sin(67.4)}{2} = 2.31 ft$$

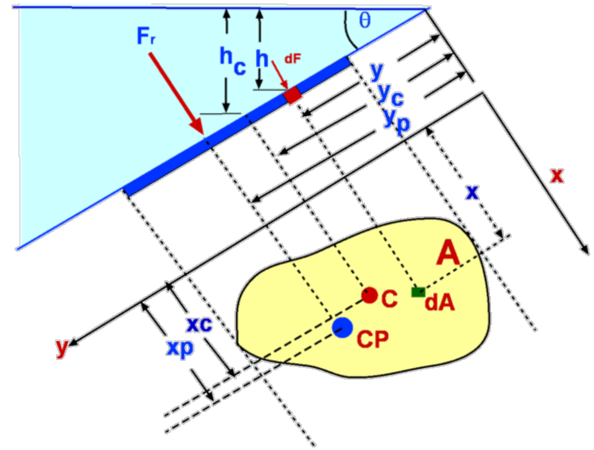
$$F_r = \left(62.4\frac{lbf}{ft^3}\right)(2.31 ft)(10 ft \bullet 5 ft) = 7207 lbf$$





LOCATION

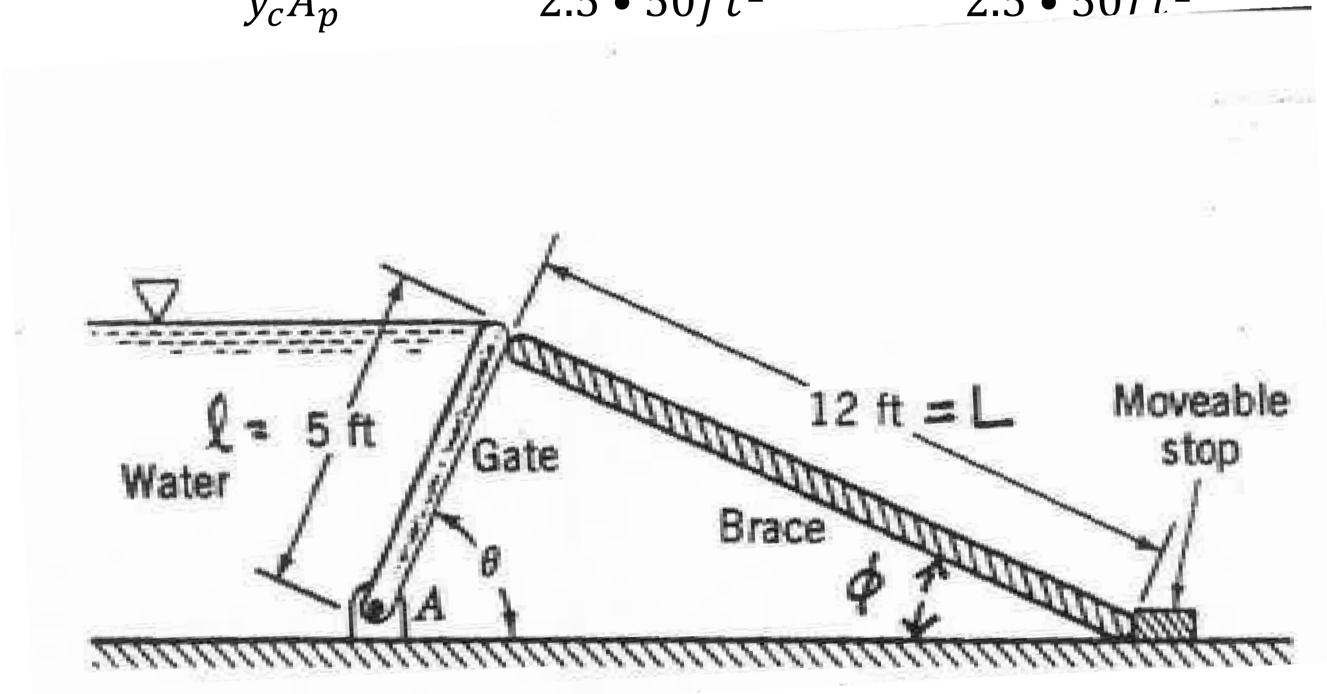
"yr" ("yp") measured from SURFACE along AXIS of plate to LINE of ACTION of resultant pressure FORCE.



$$LOCATION$$

$$y_c = \frac{l}{2} = 2.5'$$

$$y_r = y_c + \frac{Ixc}{y_c A_p} = 2.5' + \frac{\frac{1}{12} bh^3}{2.5 \bullet 50 ft^2} = 2.5' + \frac{\frac{1}{12} 10 \bullet 5^3}{2.5 \bullet 50 ft^2} = 3.3'$$



STATIC EQUILIBRIUM FBD

BRACE

$$\sum_{A}^{CCW+} M = 0$$

$$+F_b l - mg\cos\theta \frac{l}{2} - F_r(l - y_r) = 0$$

$$F_b = \frac{mg\cos\theta \frac{l}{2} + F_r(l - y_r)}{l}$$

HINGE

$$\sum_{x}^{\rightarrow} F = 0$$

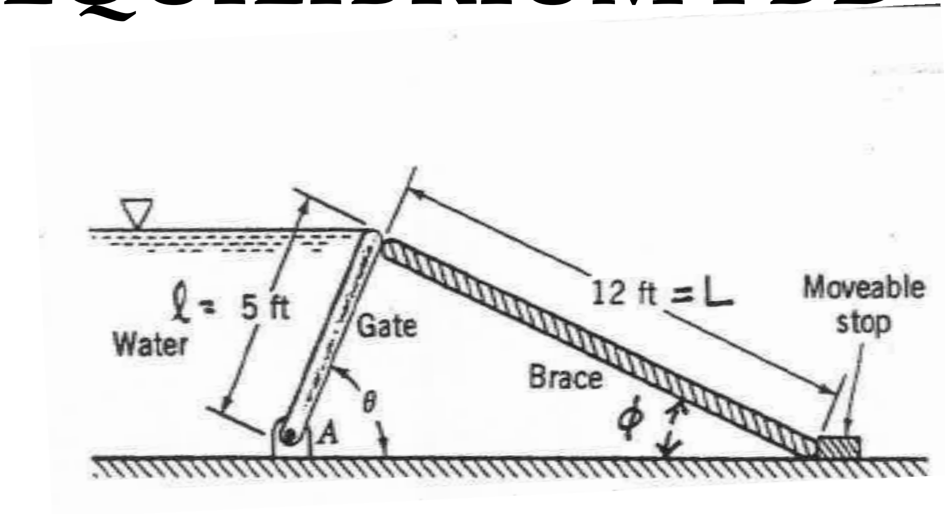
$$A_x + F_r \sin\theta - F_b \cos\phi = 0$$

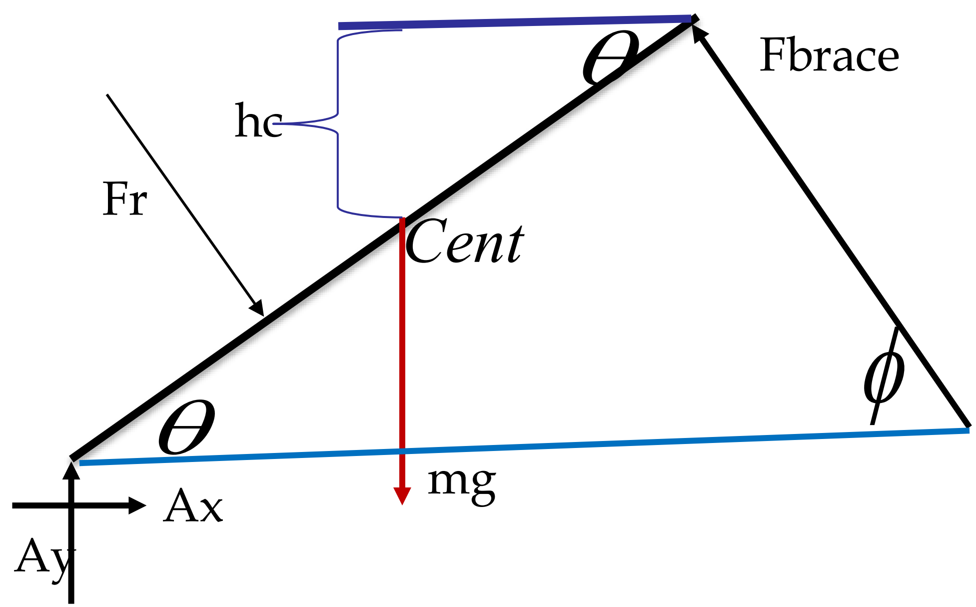
$$A_x = F_b \cos\phi - F_r \sin\theta$$

$$\uparrow \sum_{y} F = 0$$

$$A_y - F_r \cos\theta + F_b \sin\phi - mg = 0$$

$$A_y = F_r \cos\theta - F_b \sin\phi + mg$$



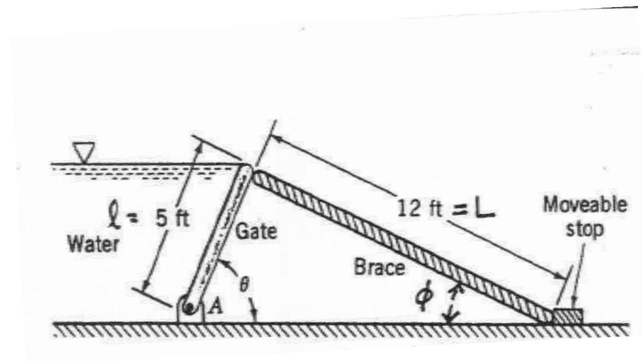


$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

$$\left\| \vec{A} \right\| = \sqrt{A_x^2 + A_y^2}$$

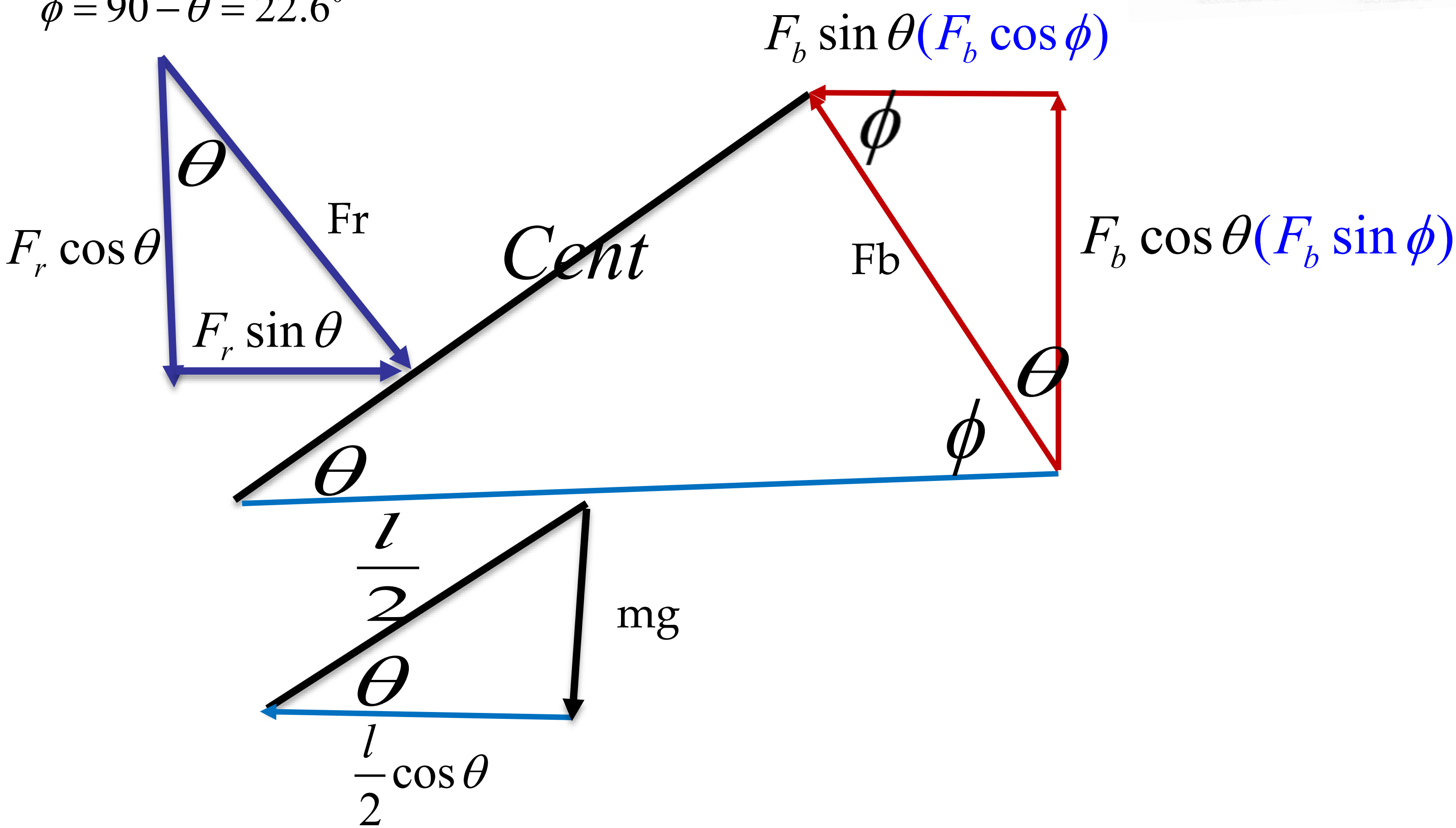
$$\theta_x = \tan^{-1} \frac{A_y}{A_x}$$

FORCE RESOLUTION



$$\theta = \tan^{-1}\frac{12}{5} = 67.4^0,$$

$$\phi = 90 - \theta = 22.6^0$$



Class 05: Hydrostatic Force on a Submerged Surface

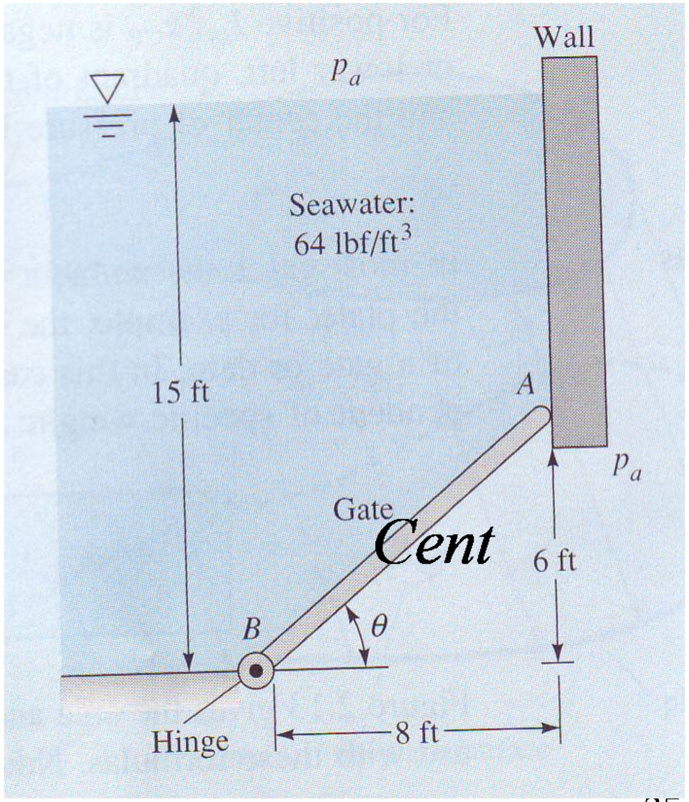
Problem # 1: **A gate, shown in Figure below is 5 ft wide, is hinged at point B, and rests against a smooth wall at point A. Compute (a) the force on the gate due to seawater pressure, (b) the horizontal normal force P exerted by the wall at point A, and (c) the reactions at the hinge B.**

$$\sin\theta = \frac{6}{L} \rightarrow L = \frac{6}{\sin\theta} = 10'$$

The gate area is $10\,ft \times 5\,ft = 50\,ft^2$

"hc" measured from SURFACE VERTICALLY to location of PLATE CENTROID.

$$F = P_C A = \gamma_{H_2 0} h_C A$$

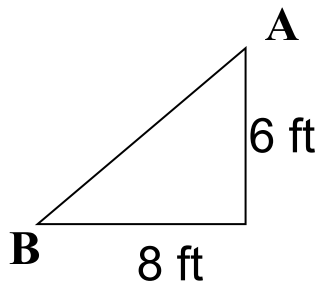


“yr” measured from SURFACE along AXIS of plate to LINE of ACTION of resultant pressure FORCE.

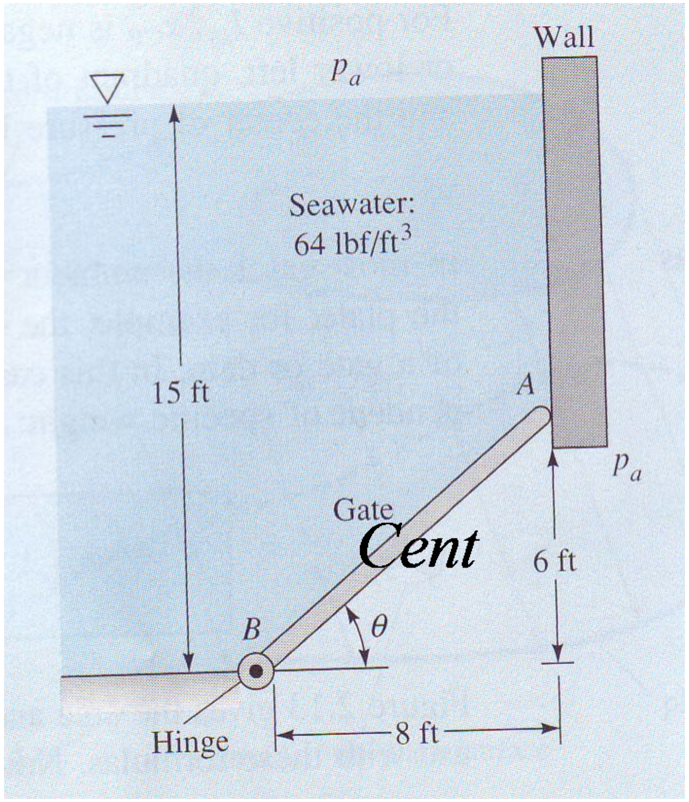
“hc” measured from SURFACE VERTICALLY to location of PLATE CENTROID.

Solution: (a) By geometry the gate is rectangular, 10 ft long (see Fig. below) from A to B and 5 ft wide.

$$\theta = 37^0 = \tan^{-1}(6/8)$$



The centroid of the gate is halfway between or at elevation 3 ft above point B.



Thus the depth h_c = 15-3 =12ft

$$\sin\theta = \frac{6}{L} \rightarrow L = \frac{6}{\sin\theta} = 10'$$

Neglect P_a as acting on both sides of the gate.

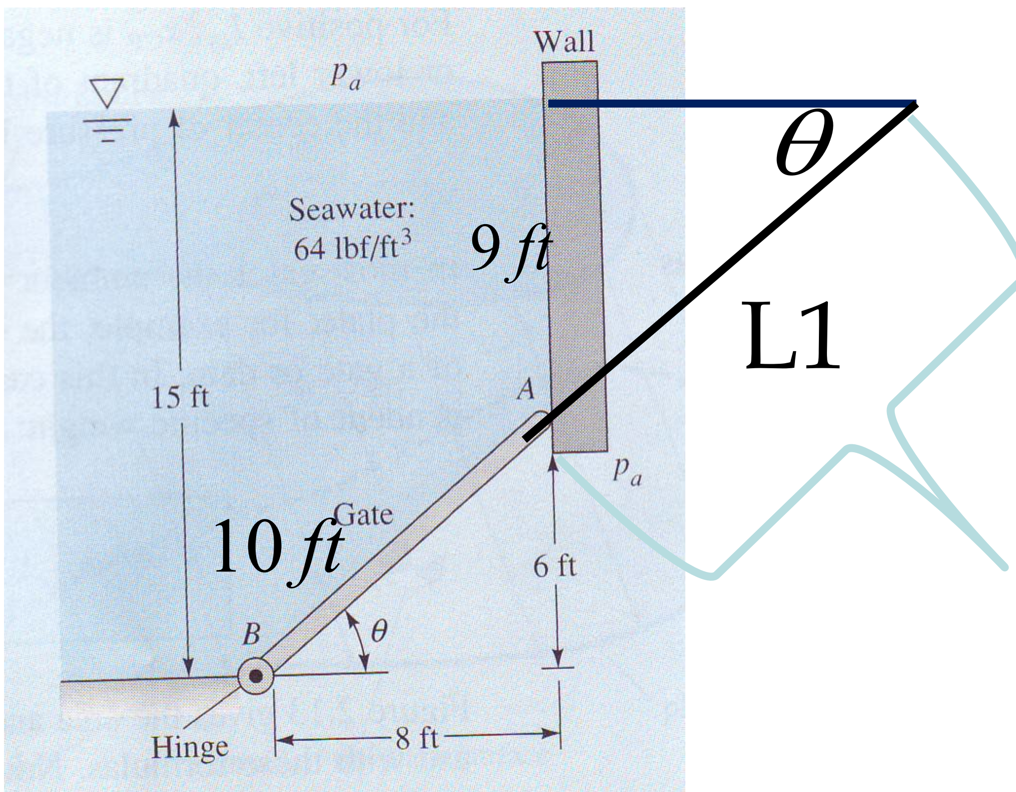
The hydrostatic Force on the gate is

$$F = P_C A = \gamma_{H_20} h_C A = \left(64\, lbf/ft^3\right)\left(12\, ft\right)\left(50\, ft^2\right) = 38,400 lbf$$

“yc” measured from SURFACE along AXIS of PLATE, to CENTROID.

$$\sin\theta = \frac{9}{L_1} \rightarrow L_1 = \frac{9}{\sin\theta} = 15'$$

$$y_c = 15' + \frac{l}{2} = 15' + \frac{10'}{2} = 20$$



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(b) We must find the center of pressure (CP) of F. A free-body diagram of the gate is shown below. The gate is a rectangle, hence

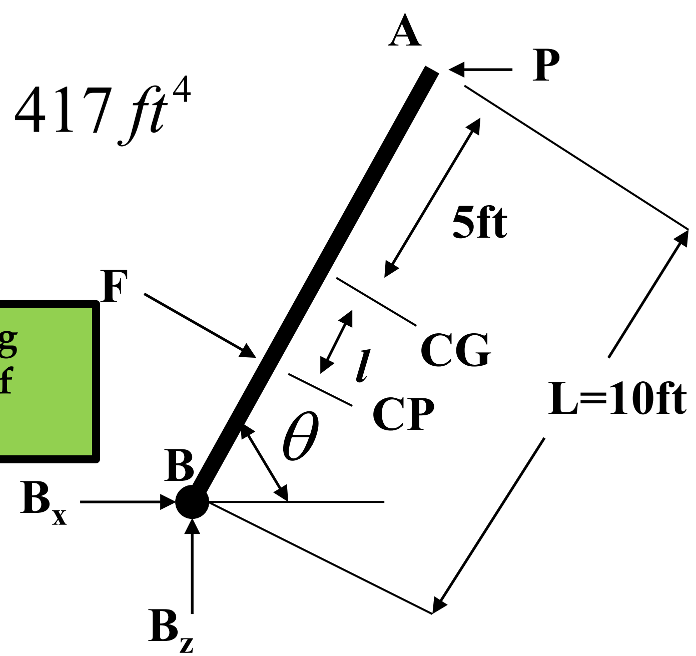
$$I_{xy} = 0 \quad and \; I_{xc} = \frac{bL^3}{12} = \frac{(5\,ft)(10\,ft)^3}{12} = 417\,ft^4$$

The distance *l* from the CG to the CP is:

$$y_R = \frac{I_{xc}}{y_c A} + y_c$$

"yr" measured from SURFACE along AXIS of plate to LINE of ACTION of resultant pressure FORCE.

$$l = y_r - y_c$$



$$Distance\ from\ point\ B\ to\ force\ F\ is:$$

$$10' + 15' - y_r = 4.583'$$

SUMMING THE MOMENTS COUNTERCLOCKWISE ABOUT B GIVES:

$$\sum_{CCW+} M_B = 0 = P \bullet 6\,ft - F(4.583') = P(6\,ft) - (38{,}400 lbf)(4.583\,ft)$$

$$\Rightarrow P = 29{,}300 lbf$$

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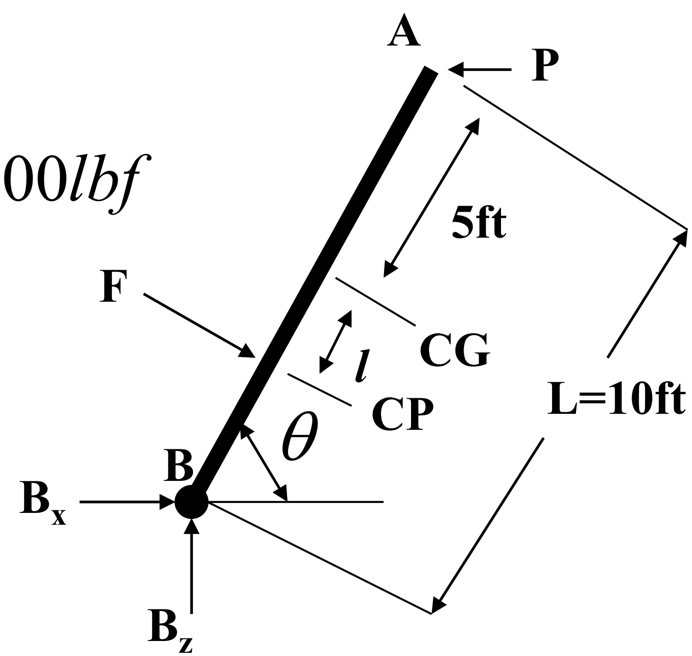
(c) With F and P known, the reactions B_x and B_z are found by summing the forces on the gate.

$$\rightarrow \sum F_x = 0 = B_x + F\,sin\,\theta - P$$

$$\Rightarrow 0 = B_x + (38,400lbf)\left(\frac{6}{10}\right) - 29,300lbf$$

$$\Rightarrow B_x = 6300lbf$$



Similarly,

$$\uparrow \sum F_z = 0 = B_z - F\,cos\,\theta$$

$$\Rightarrow 0 = B_z - (38,400lbf)\left(\frac{8}{10}\right)$$

$$\Rightarrow B_z = 30,700lbf$$

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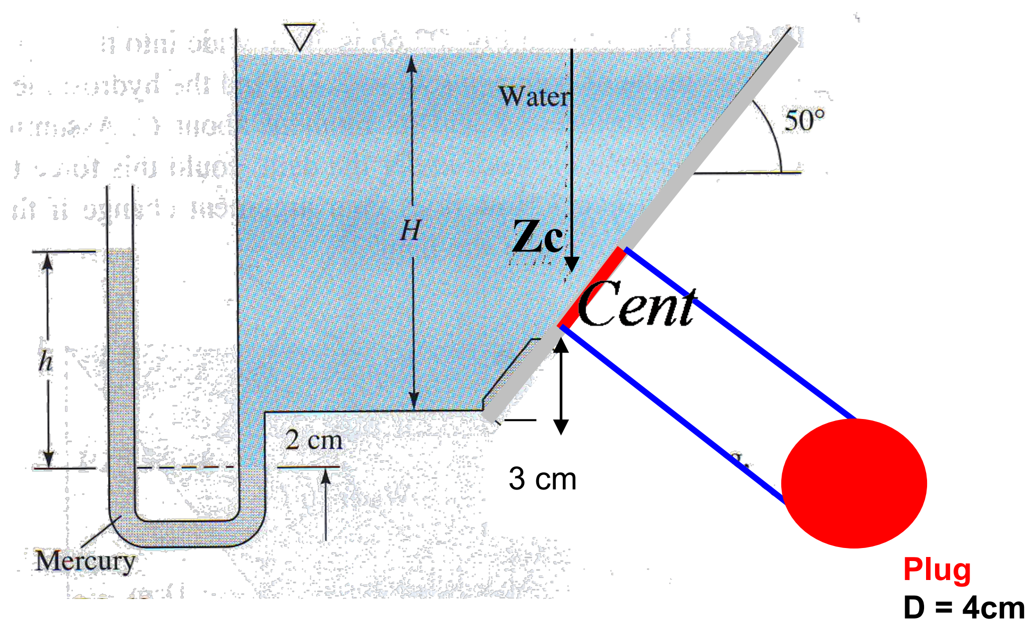
Problem # 2: **A reservoir filled with water has a 4cm diameter circular plug at 3cm above the bottom of right corner. The plug will pop out if the hydrostatic force acting on it exceeds 30N. For this condition:**

a. What will be the pressure at the bottom of the reservoir?

b. What will be the reading, *h*, on the mercury manometer on the left side?

Solution:



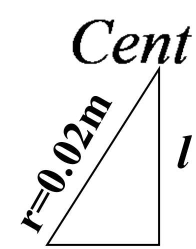
$$F_R = \gamma_{H_2O} Z_C A$$

$$\Rightarrow 30N = \left(9800\frac{N}{m^3}\right) Z_C \left(\frac{\pi}{4}(0.04m)^2\right)$$

$$\Rightarrow Z_C = 2.44m$$

$$l = r\,sin\,50^o = 0.02m \times sin\,50^o = 0.015m$$

$$H = Z_C + l + 0.03m = 2.44m + 0.015m + 0.03m = 2.49m$$



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(a) Pressure at the bottom:

$$P_1 = \gamma_{H_2O} H = 9800 \frac{N}{m^3} \times (2.49m) = 24,402 Pa = 24.402 kPa$$

(b) Use Manometry:

$$\gamma_{Hg} = 13.55 \bullet \gamma_{H_2O} = 132,790 \frac{N}{m^3}$$

$$P_1 + \gamma_{H_2O} \times (\Delta z = 0.02m) - \gamma_{Hg} \times h = P_0$$

$$h = \frac{P_1 + \gamma_{H_2O} \times (\Delta z = 0.02m)}{132,790 \frac{N}{m^3}}$$

2

$$\Rightarrow h = 0.19m$$