MOMENTUM & MATRIX STUDY AID

MECH-322 Fluid Mechanics

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ASME FELLOW

Engineering Solutions are no Better Than Engineering Communications

Only through a commitment to a learning process can one expand the mind, and think farther and deeper than ever imagined possible.

1. Water flows through the branching pipe shown without friction. Determine:

- Pressure at 1, 2 and 3 if external reaction forces to restrain branch pipe is RX = 1200N. Ignore weight of water.

- Setup simulation matrix of the form:

$$[A]\{\phi\} = \{F\}$$

where is unknow system parameters of with driving input force parameters of:

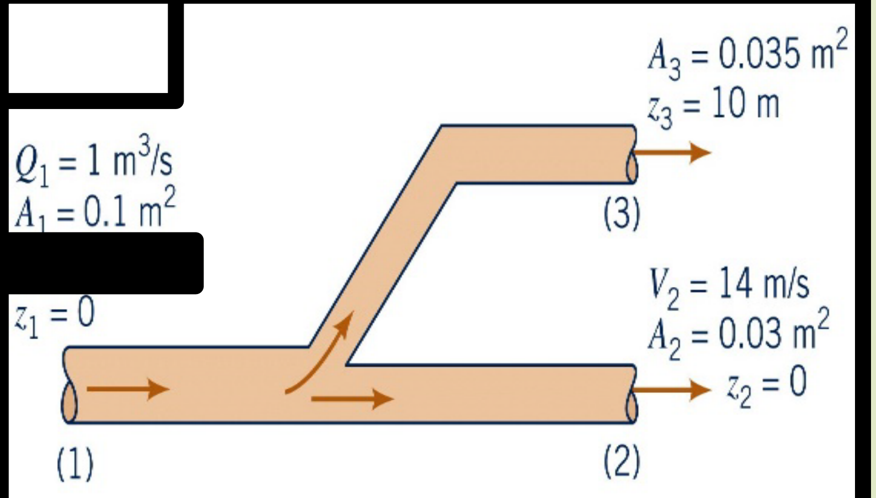


Figure P3.102

MASS CONTINUITY

$$V_1 = \frac{Q_1}{A_1} = 10\frac{m}{s}$$

$$Q_2 = V_2 A_2 = 0.42\frac{m^3}{s}$$

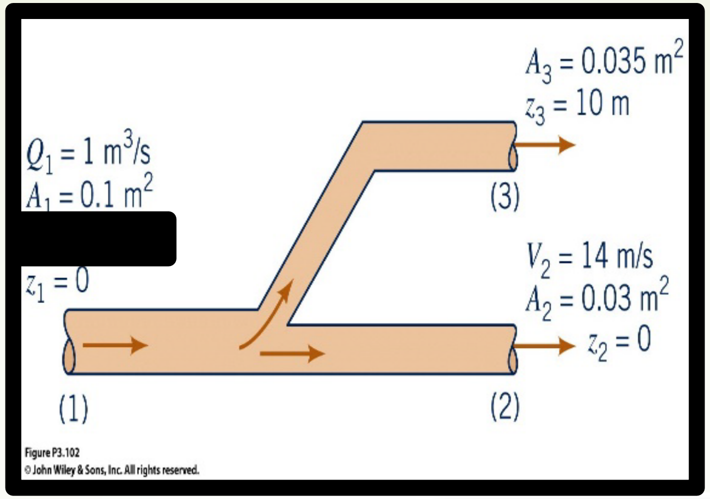
$$\text{STEADY STATE} \rightarrow \frac{d}{dt}\left[\overbrace{\rho\forall}^{mass}\right] = 0$$

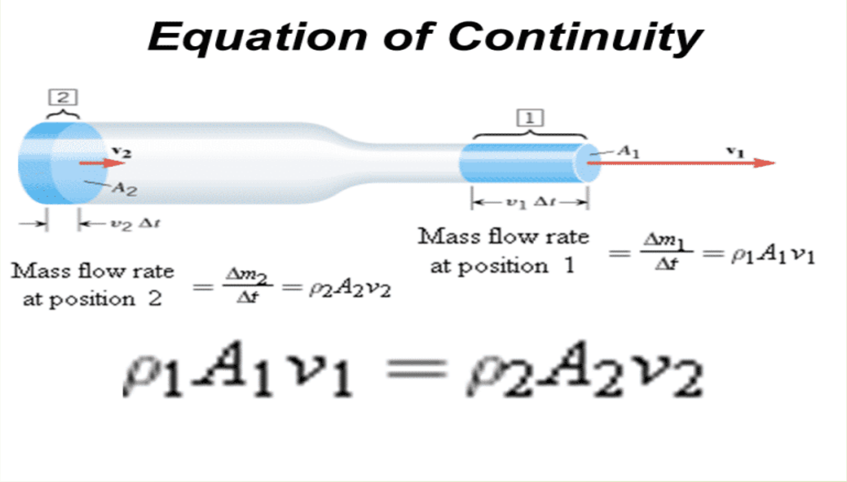
$$\sum_{in} Q = \sum_{out} Q$$

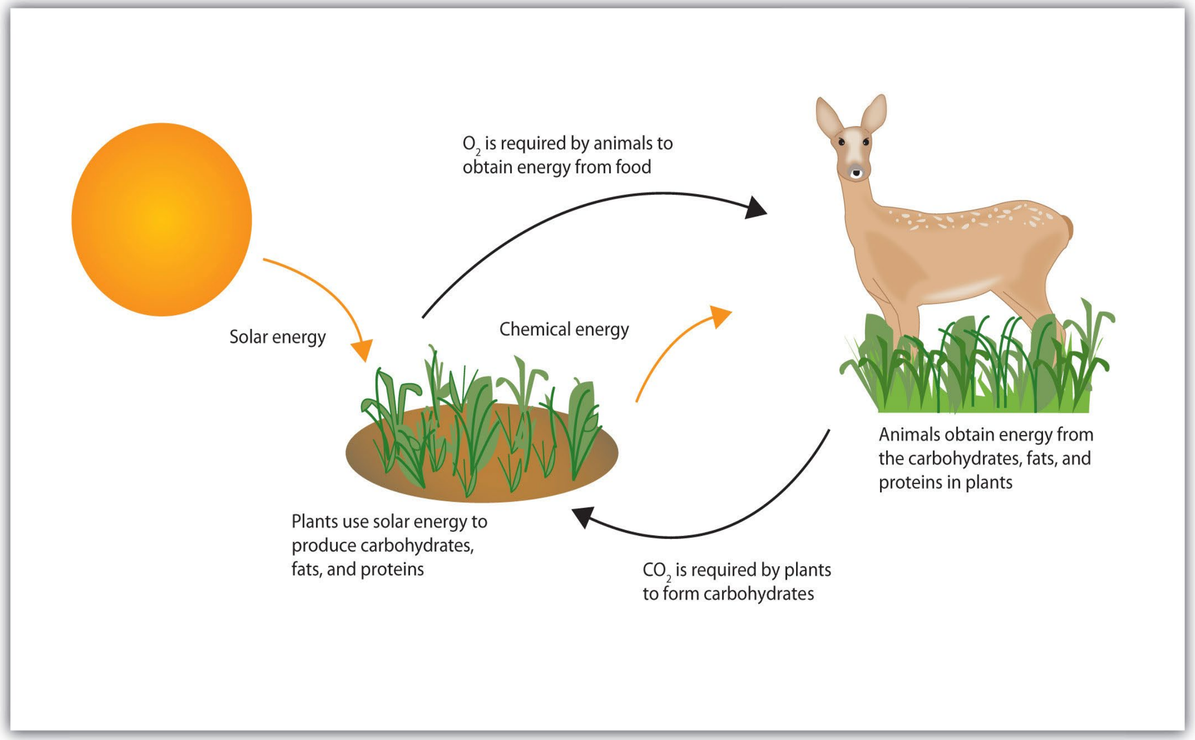
$$Q_1 = Q_2 + Q_3$$

$$Q_3 = Q_1 - Q_2 = 1 - 0.42 = 0.58\frac{m^3}{s}$$

$$V_3 = \frac{Q_3}{A_3} = 16.57\frac{m}{s}$$







BERNOULLI

BERNOULLI 1-2 (EQN. 1)

$$\dot{m}_1\left(\frac{P_1}{\gamma_f}+\frac{V_1^2}{2g}+z_1\right)=\dot{m}_2\left(\frac{P_2}{\gamma_f}+\frac{V_2^2}{2g}+z_2\right)$$

$$P_1\frac{\overbrace{\dot{m}_1}^{a_{11}}}{\gamma_f}-P_2\frac{\overbrace{\dot{m}_2}^{a_{12}}}{\gamma_f}=\overbrace{\dot{m}_2\left(\frac{V_2^2}{2g}+z_2\right)-\dot{m}_1\left(\frac{V_1^2}{2g}+z_1\right)}^{F_1}$$

BERNOULLI 1-3 (EQN. 2)

$$\dot{m}_1\left(\frac{P_1}{\gamma_f}+\frac{V_1^2}{2g}+z_1\right)=\dot{m}_3\left(\frac{P_3}{\gamma_f}+\frac{V_3^2}{2g}+z_3\right)$$

$$P_1\frac{\overbrace{\dot{m}_1}^{a_{21}}}{\gamma_f}-P_3\frac{\overbrace{\dot{m}_3}^{a_{23}}}{\gamma_f}=\overbrace{\dot{m}_3\left(\frac{V_3^2}{2g}+z_3\right)-\dot{m}_1\left(\frac{V_1^2}{2g}+z_1\right)}^{F_2}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_3 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}=\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

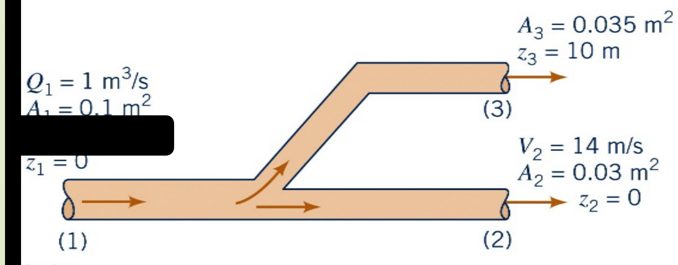


Figure P3.102

MOMENTUM

$$\overset{\leftarrow +}{\sum} F_x = \cancel{\frac{dM_{CV}}{dt}} + \sum_{out} (u_{out} \pm)\dot{m}_{out} - \sum_{in} (u_{in} \pm)\dot{m}_{in}$$

$$-R_X - P_1 A_1 + P_2 A_2 + P_3 A_3 = 0 + V_3(-)\dot{m}_3 + V_2(-)\dot{m}_2 - V_1(-)\dot{m}_1$$

$$P_1 \overbrace{(-A_1)}^{a_{31}} + P_2 \overbrace{A_2}^{a_{32}} + P_3 \overbrace{A_3}^{a_{33}} = \overbrace{R_x - (V_3\dot{m}_3 + V_2\dot{m}_2) + V_1\dot{m}_1}^{F_3}$$

+X AXIS ←

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

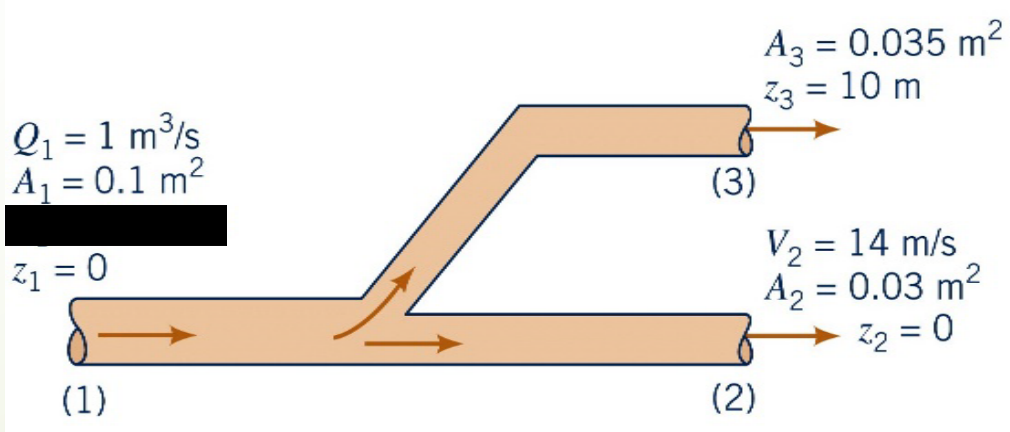


Figure P3.102

Rx →

$$\begin{bmatrix} \frac{\dot m_1}{\gamma_f} & -\frac{\dot m_2}{\gamma_f} & 0 \\ \frac{\dot m_1}{\gamma_f} & 0 & -\frac{\dot m_3}{\gamma_f} \\ -A_1 & A_2 & A_3 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} F_1 = \dot m_2\left(\frac{V_2^2}{2g} + z_2\right) - \dot m_1\left(\frac{V_1^2}{2g} + \overbrace{z_1}^{0}\right) \\ F_2 = \dot m_3\left(\frac{V_3^2}{2g} + \overbrace{z_3}^{0}\right) - \dot m_1\left(\frac{V_1^2}{2g} + \overbrace{z_1}^{0}\right) \\ F_3 = R_x - (V_3\dot m_3 + V_2\dot m_2) + V_1\dot m_1 \end{Bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

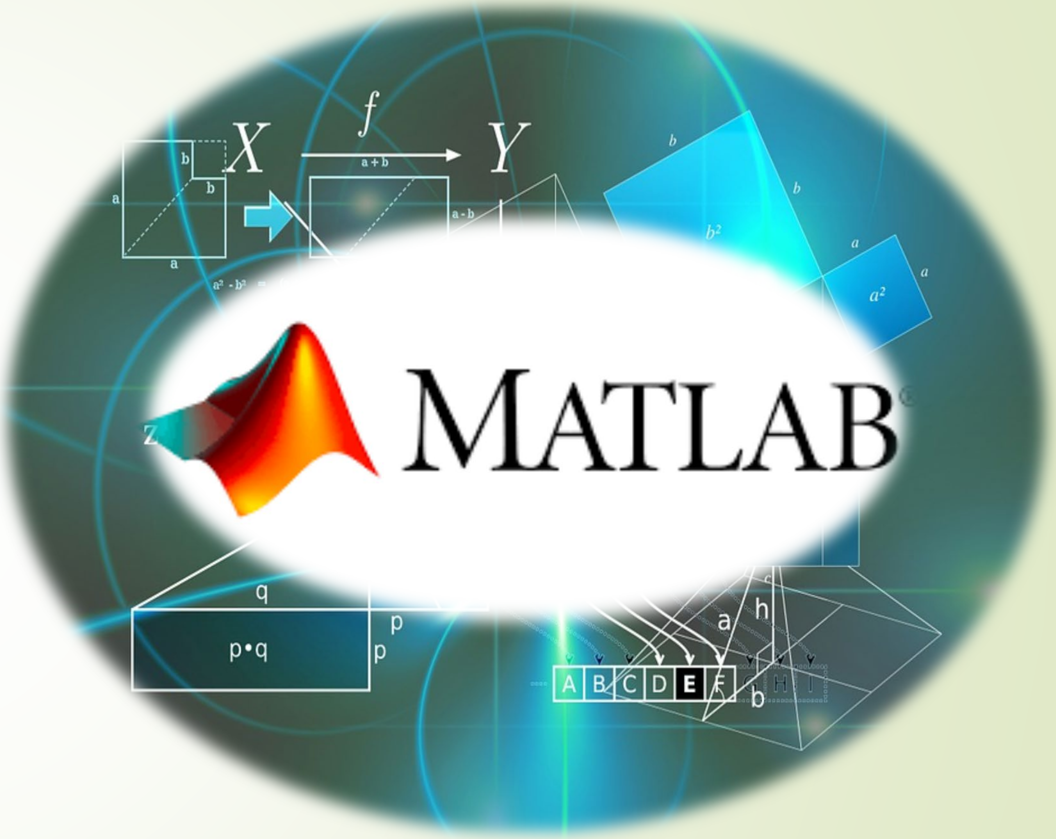
DRIVING INPUT PARAMETERS

Q_1, Q_2, R_x

$Q_1 > Q_2$

MATRIX SOLUTION METHODS

$$\begin{bmatrix} a_{11} & a_{12} & a_3 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$



$$\underbrace{\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}}_{3x1} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_3 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1}}_{3x3=\text{INVERSE MATRIX}} \underbrace{\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}}_{3x1}$$

CRAMER's RULE
"...poor man's matrix solver"

- **Cramer's rule** is an explicit formula for the solution of a system of linear equations with as many equations as unknowns, valid whenever the system has a unique solution. It expresses the solution in terms of the determinants of the (square) coefficient matrix and of matrices obtained from it by replacing one column by the column vector of right-sides of the equations. It is named after Gabriel Cramer, who published the rule for an arbitrary number of unknowns in **1750**,[1][2]

CRAMER's RULE
3x3 Example

Consider 3x3 Matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

$where:$

$| \; | \rightarrow$ DETERMINATE != 0

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

VERY GOOD FOR EMBEDDING IN INDEPENDENT SOLUTION CODES

OLD SCHOOL BRUTE FORCE SOLUTION

USE EQN. 1 BERNOULLI) to solve for P1

USE EQN. 2 BERNOULLI) to solve for P3

$$P_1 = \frac{F_1 + P_2 a_{12}}{a_{11}}$$

$$P_3 = \frac{F_2 - P_1 a_{21}}{-a_{23}} = \frac{F_2 - \left(\frac{F_1 + P_2 a_{12}}{a_{11}}\right) a_{21}}{-a_{23}}$$

BERNOULLI 1-2 (EQN. 1)

$$\dot{m}_1\left(\frac{P_1}{\gamma_f} + \frac{V_1^2}{2g} + z_1\right) = \dot{m}_2\left(\frac{P_2}{\gamma_f} + \frac{V_2^2}{2g} + z_2\right)$$

$$P_1 \overbrace{\frac{\dot{m}_1}{\gamma_f}}^{a_{11}} - P_2 \overbrace{\frac{\dot{m}_2}{\gamma_f}}^{a_{12}} = \overbrace{\dot{m}_2\left(\frac{V_2^2}{2g} + z_2\right) - \dot{m}_1\left(\frac{P_1}{\gamma_f} + z_1\right)}^{F_1}$$

BERNOULLI 1-3 (EQN. 2)

$$\dot{m}_1\left(\frac{P_1}{\gamma_f} + \frac{V_1^2}{2g} + z_1\right) = \dot{m}_3\left(\frac{P_3}{\gamma_f} + \frac{V_3^2}{2g} + z_3\right)$$

$$P_1 \overbrace{\frac{\dot{m}_1}{\gamma_f}}^{a_{21}} - P_3 \overbrace{\frac{\dot{m}_3}{\gamma_f}}^{a_{23}} = \overbrace{\dot{m}_3\left(\frac{V_3^2}{2g} + z_3\right) - \dot{m}_1\left(\frac{P_1}{\gamma_f} + z_1\right)}^{F_2}$$

USE EQN. 3 (MOMENTUM) to solve for P2

$$P_2 = \frac{F_3 + P_1 A_1 - P_3 A_3}{A_2} = \frac{F_3 + \left(\frac{F_1 + P_2 a_{12}}{a_{11}}\right) A_1 - \left(\frac{F_2 - \left(\frac{F_1 + P_2 a_{12}}{a_{11}}\right) a_{21}}{-a_{23}}\right) A_3}{A_2}$$

$$P_2 \left(1 - \frac{a_{12} A_1}{a_{11} A_2} + \frac{a_{12} a_{21} A_3}{A_2 a_{11} a_{23}}\right) = \frac{F_3 + \left(\frac{F_1}{a_{11}}\right) A_1 - \left(\frac{F_2 - \left(\frac{F_1}{a_{11}}\right) a_{21}}{-a_{23}}\right) A_3}{A_2}$$

COMBINE LIKE TERMS

$$\overset{\leftarrow +}{\sum} F_x = \frac{dM_{CV}}{dt} + \sum_{out} (u_{out} \pm) \dot{m}_{out} - \sum_{in} (u_{in} \pm) \dot{m}_{in}$$

$$-R_X - P_1 A_1 + P_2 A_2 + P_3 A_3 = 0 + V_3 (-) \dot{m}_3 + V_2 (-) \dot{m}_2 - V_1 (-) \dot{m}_1$$

$$P_1 \overbrace{(-A_1)}^{a_{31}} + P_2 \overbrace{A_2}^{a_{32}} + P_3 \overbrace{A_3}^{a_{33}} = \overbrace{R_x - (V_3 \dot{m}_3 + V_2 \dot{m}_2) + V_1 \dot{m}_1}^{F_3}$$

COMBINE LIKE TERMS

$$P_2\left(1-\frac{a_{12}A_1}{a_{11}A_2}+\frac{a_{12}a_{21}A_3}{A_2a_{11}a_{23}}\right)=\frac{F_3+\left(\frac{F_1}{a_{11}}\right)A_1-\left(\frac{F_2-\left(\frac{F_1}{a_{11}}\right)a_{21}}{-a_{23}}\right)A_3}{A_2}$$

$$P_2=\frac{\dfrac{F_3+\left(\frac{F_1}{a_{11}}\right)A_1-\left(\frac{F_2-\left(\frac{F_1}{a_{11}}\right)a_{21}}{-a_{23}}\right)A_3}{A_2}}{\left(1-\frac{a_{12}A_1}{a_{11}A_2}+\frac{a_{12}a_{21}A_3}{A_2a_{11}a_{23}}\right)}$$

$$\overset{\leftarrow +}{\sum}F_x=\frac{dM_{CV}}{dt}+\sum_{out}(u_{out}\pm)\dot{m}_{out}-\sum_{in}(u_{in}\pm)\dot{m}_{in}$$

$$-R_X-P_1A_1+P_2A_2+P_3A_3=0+V_3(-)\dot{m}_3+V_2(-)\dot{m}_2-V_1(-)\dot{m}_1$$

$$P_1\overbrace{(-A_1)}^{a_{31}}+P_2\overbrace{A_2}^{a_{32}}+P_3\overbrace{A_3}^{a_{33}}=\overbrace{R_x-(V_3\dot{m}_3+V_2\dot{m}_2)+V_1\dot{m}_1}^{F_3}$$

FINAL RESULT

$$P_2\left[\frac{N}{m^2}\right]=\frac{F_3+\left(\frac{F_1}{a_{11}}\right)A_1-\left(\frac{F_2-\left(\frac{F_1}{a_{11}}\right)a_{21}}{-a_{23}}\right)A_3}{A_2\left(1-\frac{a_{12}A_1}{a_{11}A_2}+\frac{a_{12}a_{21}A_3}{A_2a_{11}a_{23}}\right)}$$

$$P_3\left[\frac{N}{m^2}\right]=\frac{F_2-\left(\frac{F_1+P_2a_{12}}{a_{11}}\right)a_{21}}{-a_{23}}$$

$$P_1\left[\frac{N}{m^2}\right]=\frac{F_1+P_2a_{12}}{a_{11}}$$